ECE 438 Lecture 2/20/2023

Announcements

1. Office hours today 2:30 pm EST
   4:00 pm EST

2. HW #4 due Wednesday

DTFT $X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$

Problems

1. signal length is infinite
2. ω is continuous

Solution

1. truncate signal
2. discretize ω

Forward DFT $X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi k n}{N}}$ (1) $\quad \omega_k = \frac{2\pi k}{N}$

Inverse transform (DTFT)

$$x[n] = \frac{1}{2\pi}\int_0^{2\pi} X(\omega) e^{j\omega n} d\omega \quad (2)$$

can approximate Eq. (2) by a sum. But will it be a true inverse transform

Consider $\sum_{k}\left\{\left(\sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi k n}{N}}\right) e^{j\frac{2\pi k m}{N}}\right\}$

Switch order of summations:

Module 1.6.1

Consider $\sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi k m}{N}}$

plug in Eq. (2)

plugging (1) from above, get

$$\sum_{k} \left\{ \sum_{n} x[n] e^{-j\frac{2\pi k n}{N}} \right\} e^{j\frac{2\pi k m}{N}}$$

$$\sum_{n=0}^{N-1} x[n] \underbrace{\sum_{k=0}^{N-1} e^{-j\frac{2\pi k(n-m)}{N}}}$$

$$\frac{1-e^{-j2\pi(n-m)}}{1-e^{-j\frac{2\pi(n-m)}{N}}} = N\delta[(n-m) \bmod N]$$

numerator is always 0 because

$$e^{-j2\pi(n-m)} = 1$$

So the only way we can something $\neq 0$ is for the denominator $=0$. This happens when $(n-m) = lN$ for some integer l. Then we have $\frac{0}{0}$!

Use Taylor series for $e^{x} = 1 + x + \ldots$

have $\lim_{n-m \to 0} \left\{ \dfrac{1-e^{-j2\pi(n-m)}}{1-e^{-j2\pi(n-m)/N}} \right\} = \dfrac{j2\pi(n-m)}{j2\pi(n-m)/N}$

$= N$!

have $\sum_{n=0}^{N-1} x(n)\, N\, \delta[(m-n) \bmod N] =$

$N x[m]$ for $\le m \le N-1$

$\Rightarrow \quad x(n) = \dfrac{1}{N} \sum_{k=0}^{N-1} X(k] e^{j\frac{2\pi k n}{N}}$ inverse DFT

is an <u>exact</u> inverse transform!

DFT pair is:

$X[k] = \sum_{n=0}^{N-1} x(n] e^{-j2\pi k n/N}$ Forward

$x[n] = \dfrac{1}{N} \sum_{k=0}^{N-1} X(k] e^{j2\pi k n/N}$ Inverse

DFT Transform properties & pairs

(1) $a_1 x_1[n] + a_2 x_2[n] \xrightarrow{DFT} a_1 X_1[k] + a_2 X_2[k]$

① [illegible]

linearity

② shifting

$$x[n-n_0] \xrightarrow{DFT} X[k]e^{\frac{-j2\pi k n_0}{N}}$$

digression

effectively $x[n] = x[n+lN]$

for integer l

and $X[k] = X[k+lN]$

for integer l

end of digression

③ modulation

$$e^{\frac{j2\pi k_0 n}{N}} x[n] \xrightarrow{DFT} X[k-k_0]$$

here $\omega_0 = \frac{2\pi k_0}{N}$, it has to have this form

For $e^{j\omega_0 n} x[n]$, can still derive a relationship to $X[k]$, but it is more complicated

④ reciprocity: If $x[n] \xrightarrow{DFT} X[k]$, then

$x[n] \xrightarrow{DFT} X[k]$

⑤ Parseval's relation:

$$\sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X[k]|^2$$

⑥ Initial value

ⓐ $$\sum_{n=0}^{N-1} x[n] = X[0]$$

ⓑ $$\sum_{k=0}^{N-1} X[k] = N x[0]$$

⑦ periodicity $x[n] = x[n + lN]$

$X[k] = X[k + lN]$ for integer l

⑨ Relation to DTFT of a finite length sequence

Suppose $x[n] = 0$, unless $0 \le n \le N-1$

then $$X_{DTFT}(\omega) = \sum_{n=0}^{N-1} x[n] e^{-j\omega n}$$

$\frac{2\pi k}{N}$

So $X_{DFT}[k] = X_{DTFT}\left(\frac{2\pi k}{N}\right)$

Transform Pairs

① $x[n] = \delta[n] \quad 0 \le n \le N-1$ (note $x[n]$ is actually periodic

$$\delta[n] \xrightarrow{DFT} 1$$

② $x[n] \equiv 1, \quad 0 \le n \le N-1$

by reciprocity:

$$1 \xrightarrow{DFT} N\delta[-k] = N\delta[k], \quad 0 \le k \le N-1$$

Since $\delta[k]$ is an even function of k

③ $x[n] = e^{\frac{j2\pi k_0 n}{N}} \xrightarrow{DFT} N\delta[k-k_0]$

missing in legacy notes

by modulation properties

$0 \le k \le N-1$ & $0 \le k_0 \le N-1$

④ $x[n] = \cos\left(\frac{2\pi k_0 n}{N}\right) \xrightarrow{DFT}$

$$\frac{N}{2}\{\delta[k-k_0] + \delta[k-(N-k_0)]\}$$

'2 }

same as

$\delta[k + k_0]$

by periodicity of DFT

Note that frequency is restricted to be an integer multiple of $\frac{2\pi}{N}$

(5) $x[n] = \begin{cases} 1, & 0 \le n \le M-1 \\ 0, & \text{else} \end{cases}$ $M < N$

$\overline{M}$ is an integer

$X[k] = X_{DTFT}\left(\frac{2\pi k}{N}\right)$

$= e^{-j\frac{2\pi k}{N}\frac{(M-1)}{2}}\ \text{psinc}_M\left[\frac{2\pi k}{N}\right]$