ECE 438 lecture 2/17/2023

PFE

Last class - example with repeated pole

$$Y(z) = \frac{1+z^{-1}}{(1-\frac{1}{2}z^{-1})^2(1-z^{-1})}$$

$$Y(z) = \frac{A_1}{(1-\frac{1}{2}z^{-1})^2} + \frac{A_2}{(1-\frac{1}{2}z^{-1})} + \frac{A_3}{1-z^{-1}}$$

$$A_3 = 8 \quad A_1 = -3, \quad A_2 = -4$$

Two more PFE examples:

① $$Y(z) = \frac{N(z)}{(1-p_1z^{-1})(1-p_2z^{-1})\cdots(1-p_Nz^{-1})}$$

There N poles $p_1, p_2, \ldots, p_N$

No poles are repeated

$$y[n] = ZT^{-1}\left\{\sum_{n=1}^{N}\frac{A_n}{(1-p_n^{-1})}\right\}$$

$$= \sum_{n=1}^{N} ZT^{-1}\left\{\frac{A_n}{1-p_nz^{-1}}\right\}$$

② One pole repeated K times

$$Y(z) = \frac{N(z)}{(1-pz^{-1})^K}$$

In this case, the PFE has the form

$$Y(z) = \frac{A_{1K}}{(1-pz^{-1})^K} + \frac{A_{2K-1}}{(1-pz^{-1})^{K-1}} + \cdots$$

$$\frac{A_{l1}}{(1-pz^{-1})}$$

How to solve for $A_{1K}, A_{2K-1}, \ldots, A_{11}$?

Use residue method (see posted legacy notes)

This concludes discussion of PFE

Stability

Bounded input - bounded output stability (BIBO)

def A system is BIBO stable if every bounded input yields a bounded output, i.e.

If $\exists$ a constant $M_x < \infty$, s.t. $|x[n]| < M_x$ $\forall n$, then $\exists$ a constant $M_y < \infty$, s.t. $|y[n]| < M_y$ $\forall n$.

What does this mean for an LTI system?

We have $y[n] = \sum_{k} h[k]\, x[n-k]$

unit sample response

So $|y[n]| = \left| \sum_{k} h[k]\, x[n-k] \right| \leq \sum_{k} |h[k]\, x[n-k]|$

triangle inequality

$$= \sum_{k} |h[k]|\,|x[n-k]|$$

$$< \sum_{k} |h[k]| (M_x) \Rightarrow M_x \|h[k]\|_1 = M_y$$

$$\|h[k]\|_1 \triangleq \sum_{k} |h[k]|$$

So we conclude that an LTI system is BIBO stable $\iff \|h[k]\|_1 < \infty$

if and only if

example

$$y[n] = x[n] - x[n-1] + \frac{1}{2} y[n-1]$$

Is it BIBO stable?

$$H(z) = \frac{1 - z^{-1}}{1 - \frac{1}{2} z^{-1}}$$

assume that the system is causal, i.e. current output doesn't depend $x[k]$ for $k > n$

$y[n]$

i.e. current output doesn't depend on future inputs

$\Rightarrow$ $h[k]$ is r.s.s.

$\Rightarrow$ $ROC\{H(z)\} = \{ z : |z| > \frac{1}{2} \}$

note that $H(z) = \frac{1}{1-z^{-1}} - ZT^{-1}\left\{ z^{-1}\left(\frac{1}{1-\frac{1}{2}z^{-1}} \right) \right\}$

$$\therefore \quad h[n] = \left(\frac{1}{2}\right)^n u[n] - \frac{1}{2}^{(n-1)} u[n-1]$$

$$= \delta[n] - \left(\frac{1}{2}\right)^n u[n-1]$$

we see that $\|h[n]\|_1 < \infty$, $\therefore$ the system is stable.

Note that

$$\|h\|_1 = \sum_k |h[k]| = 1 + \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k$$

$$= 1 + \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k - 1$$

$$= \frac{1}{1-\frac{1}{2}} = 2 < \infty$$

This completes Module 1,5,6 and materials for HW#4

New topic: Discrete Fourier Transform (DFT) and FFT algorithm

This is essentially a "free lunch".

Discrete Fourier Transform (DFT)

Consider DTFT

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$x[n] = \frac{1}{2\pi} \int_{0}^{2\pi} X(\omega) e^{j\omega n} d\omega$$

note that is valid by periodicity of DTFT

Suppose we wish to perform spectral analysis for a real signal (not in sense of real vs. complex), but real in sense of an audio recording or an image (selfie)

Limitations

1. summation in $X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$ is over an infinite number of terms

2. ω is continuous-valued — have to discretize it

3. Inverse transform involves an

integral:

$$x[n] = \frac{1}{2\pi}\int_0^{2\pi} X(\omega)e^{j\omega n}\, d\omega$$

Solution

① truncate signal, $x[n] \neq 0$, only for $0 \le n \le N-1$

② discretize ω; i.e. let $\omega_k = \frac{2\pi k}{N}$, $k = 0, \ldots, N-1$

sampling ω at a uniform interval $\frac{2\pi}{N}$

③ can discretize the integral

$$\tilde{x}[n] = \underline{\frac{1}{2\pi}\sum_{k=0}^{N-1} X[k]e^{j\frac{2\pi k}{N}n}}\underline{\left(\frac{2\pi}{N}\right)}$$

effectively $d\omega$

Riemann sum approximation to the integral

Let's define

$$\boxed{X_{DFT}[k]} = \underline{\sum_{n=0}^{N-1} x[n]\, e^{-j2\pi\frac{kn}{N}}}$$ Forward DFT

What about inverse DFT?

Rather discretizing the integral, let's try something different:

Consider $\sum_{k=0}^{N-1} X_{DFT}[k]\, e^{j\frac{2\pi k m}{N}} = ?$

$$\sum_{k=0}^{N-1}\left\{\sum_{m=0}^{N-1} x[m]e^{-j2\pi\frac{km}{N}}\right\}e^{j\frac{2\pi kn}{N}}$$

What should we do next? (trick)

Ans: switch order of summations