ECE Lecture Monday 13 February

PFE - inverse ZT

example

$$x[n] = \left(\frac{1}{3}\right)^n u[n]$$

$$\underline{y[n] = x[n] - x[n-1] + \frac{1}{2} y[n-1]}$$

$$X(z) = \frac{1}{1-\frac{1}{3}z^{-1}}, \; |z| > \frac{1}{3}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1-z^{-1}}{1-\frac{1}{2}z^{-1}}$$

we have

$$Y(z) = H(z)X(z) = \frac{(1-z^{-1})}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{3}z^{-1})}$$

$$ROC\{Y(z)\} = ROC\{H(z)\} \cap ROC\{X(z)\}$$

$$= \{ z : |z| > \tfrac{1}{2} \}$$

assuming that the system is causal

$$\left\{ Y(z) = \underbrace{\frac{-3}{1-\frac{1}{2}z^{-1}}}_{Y_1(z)} + \underbrace{\frac{4}{1-\frac{1}{3}z^{-1}}}_{Y_2(z)} \right.$$

$$ROC\{Y(z)\} = ROC\{Y_1(z)\} \cap ROC\{Y_2(z)\}$$

$$= \{z : |z| > \tfrac{1}{3}\} \cap \{z : |z| > \tfrac{1}{2}\}$$

$= \{ z : |z| > \frac{1}{2} \}$

use ZT pair:

$$\underline{a^n u[n]} \xrightarrow{ZT} \frac{1}{1 - a z^{-1}}, |z| > |a|$$

$$y[n] = -3\left(\frac{1}{2}\right)^n u[n] + 4\left(\frac{1}{3}\right)^n u[n]$$

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ignore previous context

suppose we have $Y(z) = \dfrac{1 - z^{-1}}{(1 - \frac{1}{2} z^{-1})(1 - \frac{1}{3} z^{-1})}$

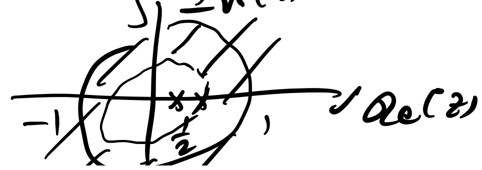
what are possible $y[n]$'s that have ZT $Y(z)$

Use the same PFE as before

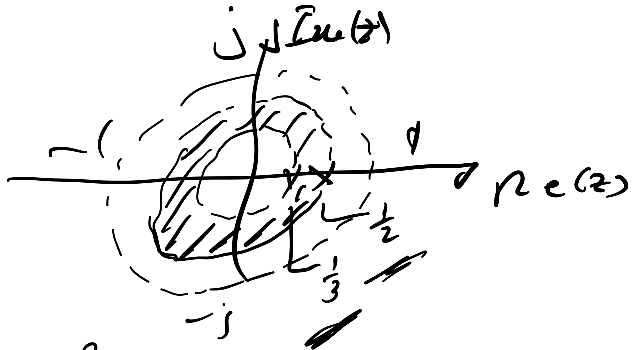
$$\boxed{Y(z) = \frac{-3}{1 - \frac{1}{2} z^{-1}} + \frac{4}{1 - \frac{1}{3} z^{-1}}}$$

There are 3 possible ROC's for this particular $Y(z)$

1. $ROC\{Y(z)\} = \{ z : |z| > \frac{1}{2} \}$ same as before



(2) $ROC\{Y(z)\} = \{z : \frac{1}{3} < |z| < \frac{1}{2}\}$



what is $y[n]$?

$$\boxed{Y(z) = \underbrace{\frac{-3}{1-\frac{1}{2}z^{-1}}}_{\text{l.s. } y_1[n]} + \underbrace{\frac{4}{1-\frac{1}{3}z^{-1}}}_{\text{r.s. } y_2[n]}}$$

same as before

l.s = left-sided

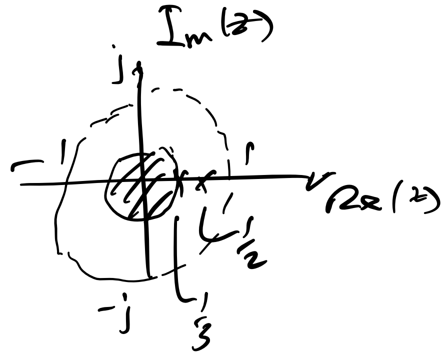
r.s = right-sided

For $y_1[n]$, use

$$-a^n u[-n-1] \xrightarrow{ZT} \frac{1}{1-az^{-1}}, \quad |z|<|a|$$

$$y[n] = \underbrace{4\left(\tfrac{1}{3}\right)^n u[n]}_{\text{same as before}} - \underbrace{3(-1)\left(\tfrac{1}{2}\right)^n u[-n-1]}_{\text{new}}$$

(3) $\text{ROC}\{Y(z)\} = \{z : |z| < \frac{1}{3}\}$



Same PFE as before

$y[n] = \underset{\text{same as before}}{y_1[n]} + \underset{\text{new}}{y_2[n]}$

Again $-a^n u[-n-1] \xrightarrow{ZT} \frac{1}{1-az^{-1}}$ $\quad |z| < |a|$

both $y_1[n]$ & $y_2[n]$ are l.s.

$$y[n] = 4(-1)\left(\frac{1}{3}\right)^n u[-n-1] + 3(-1)\left(\frac{1}{2}\right)^n u[-n-1]$$

$$Y(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \lim_{N \to \infty} \sum_{n=0}^{N-1} a^n z^{-n}$$

r.s. case

$$= \lim_{N \to \infty} \frac{1-(az^{-1})^N}{1-(az^{-1})} = \frac{1}{1-az^{-1}}, \quad |az^{-1}| < 1$$

$\Rightarrow z > |a|$

l.s. case

$$Y(z) = -\sum_{n=-\infty}^{-1} a^n z^{-n} \qquad \text{let } m = -n$$

$$= -\sum_{m=1}^{\infty} a^{-m} z^m$$

$$= -\sum_{m=0}^{\infty} a^{-m} z^m + 1$$

$$= \lim_{N \to \infty} \left(- \sum_{m=0}^{N-1} a^{-m} z^{m} \right) + 1$$

$$= \lim_{N \to \infty} - \frac{1 - a^{-N} z^{N}}{1 - a^{-1} z} + 1$$

$$Y(z) = - \underbrace{\frac{1}{1 - a^{-1} z}}_{} + 1 \quad , \text{if } |a^{-1} z| < 1$$

$$\Rightarrow |z| < a$$

$$= \left(\frac{a z^{-1}}{a z^{-1} - 1} \right) + 1 = \frac{1 - a z^{-1}}{1 - a z^{-1}}$$

$$= \frac{1}{1 - a z^{-1}}$$

If signal is l.s., ROC has to be inside a circle of some radius in z-plane

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Two other considerations for PFE

① A pole with multiplicity 2:

$$Y(z) = \frac{1 + z^{-1}}{(1 - \frac{1}{2} z^{-1})^{2} (1 - z^{-1})}$$

This is what is new,

Correct form for the PFE in:

$$Y(z) = \frac{A_1}{(1-\frac{1}{2}z^{-1})^2} + \frac{A_2}{1-\frac{1}{2}z^{-1}} + \frac{A_3}{1-z^{-1}}$$

How to solve for A_1, A_2 & A_3 ??

Step 1 multiply both sides by $(1-\frac{1}{2}z^{-1})^2(1-z^{-1})$

$$\boxed{1+z^{-1} = (1-z^{-1})A_1 + (1-\tfrac{1}{2}z^{-1})(1-z^{-1})A_2 + (1-\tfrac{1}{2}z^{-1})^2 A_3}$$

Step 2 let $z^{-1}=1$, have $2 = A_3(\frac{1}{2})^2 \Rightarrow A_3 = 8$

Step 3 let $z^{-1}=2$, $3 = (-1)A_1 \Rightarrow A_1 = -3$

Step 4 To get A_2, differentiate both sides with respect to z^{-1}:

$$1 = -A_1 + \left\{-\tfrac{1}{2}(1-z^{-1}) - (1-\tfrac{1}{2}z^{-1})\right\}A_2 + 2(1-\tfrac{1}{2}z^{-1})(-\tfrac{1}{2})A_3$$

let $z^{-1}=2 \Rightarrow 1 = -\overset{-3}{A_1} + \frac{1}{2}A_2 + A_3$

$1 = 3 + \frac{1}{2}A_2 + 8 \Rightarrow A_2 = (-10)2$

$= -20$

HW#3