ECE 438 Lecture 2/10/2023

Announcements:

To be done:

- Post solution to HW #3 (Finish reviewing it first)
- Post lecture recording from Wednesday
- Post lecture recording from today
- Address concerns about office hours & "help" session
- Post HW # 4
- Update Formula Sheet for Exam #1

How to study for Exam #1

- Review formula sheet
- Review past ~~ECE~~ 438 Exams from 1, 2, 3 previous offerings
- Review HW's 1, 2, 3
- Review this years recorded lectures

Graphical interpretation of frequency response

Example

$$y[n] = x[n] + x[n-1] - \frac{1}{2} y[n-2]$$

from legacy notes

$$H(z) = \frac{z(z+1)}{(z - j/\sqrt{2})(z + j/\sqrt{2})}$$

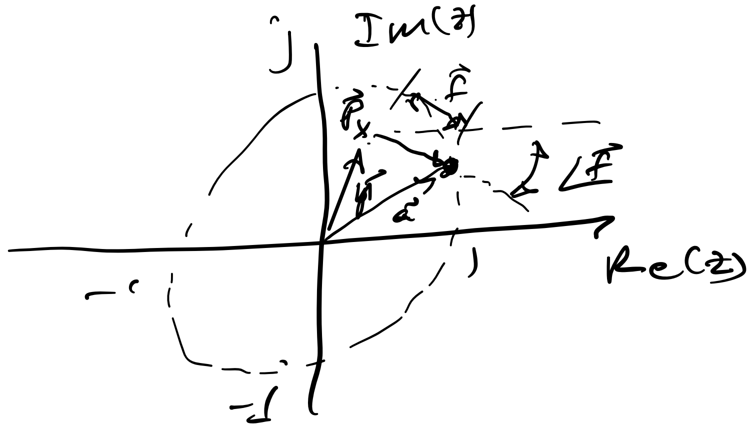
$$\left|H_{DTFT}(\omega)\right| = \frac{|e^{j\omega}||e^{j\omega}+1|}{|e^{j\omega}-j/\sqrt{2}||e^{j\omega}+j/\sqrt{2}|}$$

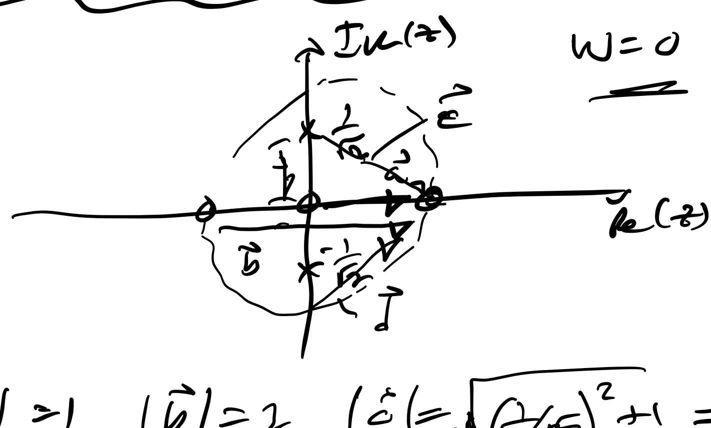
$\vec{a} = e^{j\omega}$, $\vec{b} = e^{j\omega}+1$ ⇐ zeros

$\vec{c} = e^{j\omega} - j/\sqrt{2}$ $\vec{d} = e^{j\omega} + j/\sqrt{2}$ ⇐ poles

$$|H(\omega)| = \frac{|\vec{a}||\vec{b}|}{|\vec{c}||\vec{d}|}, \quad \angle H(\omega) = \angle\vec{a} + \angle\vec{b} - (\angle\vec{c} + \angle\vec{d})$$

Contribution of a single root: $e^{j\omega} - \vec{p} = \vec{f}$





$W = 0$

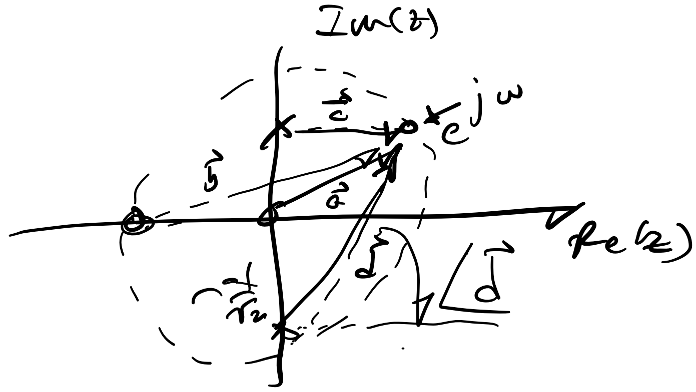
$|\vec{a}| = 1$ $|\vec{b}| = 2$ $|\vec{c}| = \sqrt{(j/\sqrt{2})^2 + 1} = \sqrt{\frac{\sqrt{2}^2+1}{(\sqrt{2})^2}}$

$= \frac{3}{\sqrt{2}}$

$\angle H(0) = 0$

$|\vec{d}| = \frac{3}{\sqrt{2}}$

$\omega = \frac{\pi}{4}$



$$|H(\pi/4)| = \frac{\sqrt{5}}{2} \approx 1.65$$

$$\angle H\left(\frac{\pi}{4}\right) = \frac{\pi}{4} + \arctan\left(\frac{\sqrt{2}}{1+\sqrt{2}}\right)$$

$$- \left(0 + \arctan\left(\frac{2/\sqrt{2}}{1/\sqrt{2}}\right)\right)$$

Rules

- If $e^{j\omega}$ is close to a pole, frequency response increases
- If $e^{j\omega}$ is close to a zero, frequency response decreases

example: $|H(\pi)| = 0$

Material is from Module 1.5.4

Module 1.5 Inverse ZT

Formally,

$$x[n] = \frac{1}{j2\pi} \oint X(z)\, z^{+n-1}\, dz$$ we will not use this

Mainly interested in rational functions of z, i.e. ratio of polynomials in z

Example

$$x[n] \longrightarrow \boxed{\begin{array}{c}\text{LTI}\\ \text{system}\end{array}} \xrightarrow{y[n]}$$

Given system equation, we want to find $y[n]$

$$x[n] = \left(\frac{1}{3}\right)^n u[n]$$

$$\underline{y[n] = x[n] - x[n-1] - \frac{1}{2} y[n-1]}$$

Options for finding $y[n]$:

1. Direct evaluation
2. Convolution of $h[n]$ & $x[n]$
3. Find inverse ZT of $\underline{H(\omega)}\,\underline{X(\omega)}$
4. ⇒ Use partial fraction expansion (PFE)

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - z^{-1}}{1 - \frac{1}{2} z^{-1}}$$

assume system is causal

Causality: $y[n]$ does not depend on future inputs

In terms of unit sample response $h[n]$, we have

$$y[n] = \sum_{k=-\infty}^{\infty} h[n-k]\,\underline{x[k]}$$

$$= \sum_{k=-\infty}^{n} h[n-k]\,x[k] \quad \text{for system to be causal}$$

this $\Rightarrow h[n-k]=0 \quad k>n \Rightarrow h[n]=0, \; n<0$

__def.__ A signal $x[n]$ is causal if $x[n]=0, \; n<0$

__right-sided__ A signal $x[n]$ is right-side if $\exists$ an integer N, such that $x[n]=0, \; \forall n<N$

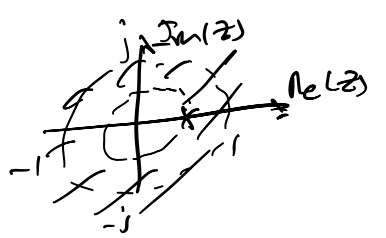
$\Rightarrow X(z) = \sum_{n=N}^{\infty} x[n] z^{-n}$ converges __outside__ a circle of some radius in the z-domain

For a causal signal, $N=0 \Rightarrow$ causal signals are a subset of right-sided signals.

Returns to our example:

$$H(z) = \frac{1-z^{-1}}{1-\frac{1}{2}z^{-1}}, \Rightarrow \text{pole at } z=\tfrac{1}{2}$$

$\Rightarrow \quad ROC\{H(z)\} = \{z : |z| > \tfrac{1}{2}\}$



Recall $x[n] = \frac{1}{3}^n \underline{u[n]}$

$$X(z) = \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n z^{-n} = \sum_{n=0}^{\infty} (3z)^{-n}$$

$$= \lim_{N \to \infty} \sum_{n=0}^{N-1} (3z)^{-n} = \lim_{N \to \infty} \frac{1-(3z)^{-N}}{1-(3z)^{-1}}$$

We want $\lim_{N \to \infty} (3z)^{-N} = 0 \Rightarrow |3z| < 1 \Rightarrow$

want $|z| > 3$ for ROC!!

$$Y(z) = H(z)X(z) = \left(\frac{1-z^{-1}}{1-\frac{1}{2}z^{-1}}\right)\left(\frac{1}{1-\frac{1}{3}z^{-1}}\right)$$

$$\text{ROC}\{Y(z)\} = \text{ROC}\{H(z)\} \cap \text{ROC}\{X(z)\}$$

$$= \{z : |z| > \tfrac{1}{2}\} \cap \{z : |z| > 3\}$$

$$\text{ROC}\{Y(z)\} = \{z : |z| > 3\}$$

Now let's look at PFE:

$$Y(z) = \frac{A_1}{1-\frac{1}{2}z^{-1}} + \frac{A_2}{1-\frac{1}{3}z^{-1}}$$

works $\underline{\underline{\text{only}}}$ if no poles are repeated

$$= \frac{1-z^{-1}}{\underbrace{\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{3}z^{-1}\right)}_{\text{multiply both sides by this}}}$$

$$(1-z^{-1}) = A_1\left(1-\frac{1}{3}z^{-1}\right) + A_2\left(1-\frac{1}{2}z^{-1}\right)$$

two equations in two unknowns:

$$\left.\begin{aligned} 1 &= A_1 + A_2 \\ -1 &= -\frac{1}{3}A_1 - \frac{1}{2}A_2 \end{aligned}\right\} \quad \begin{aligned} A_1 &= -3 \\ A_2 &= 4 \end{aligned}$$