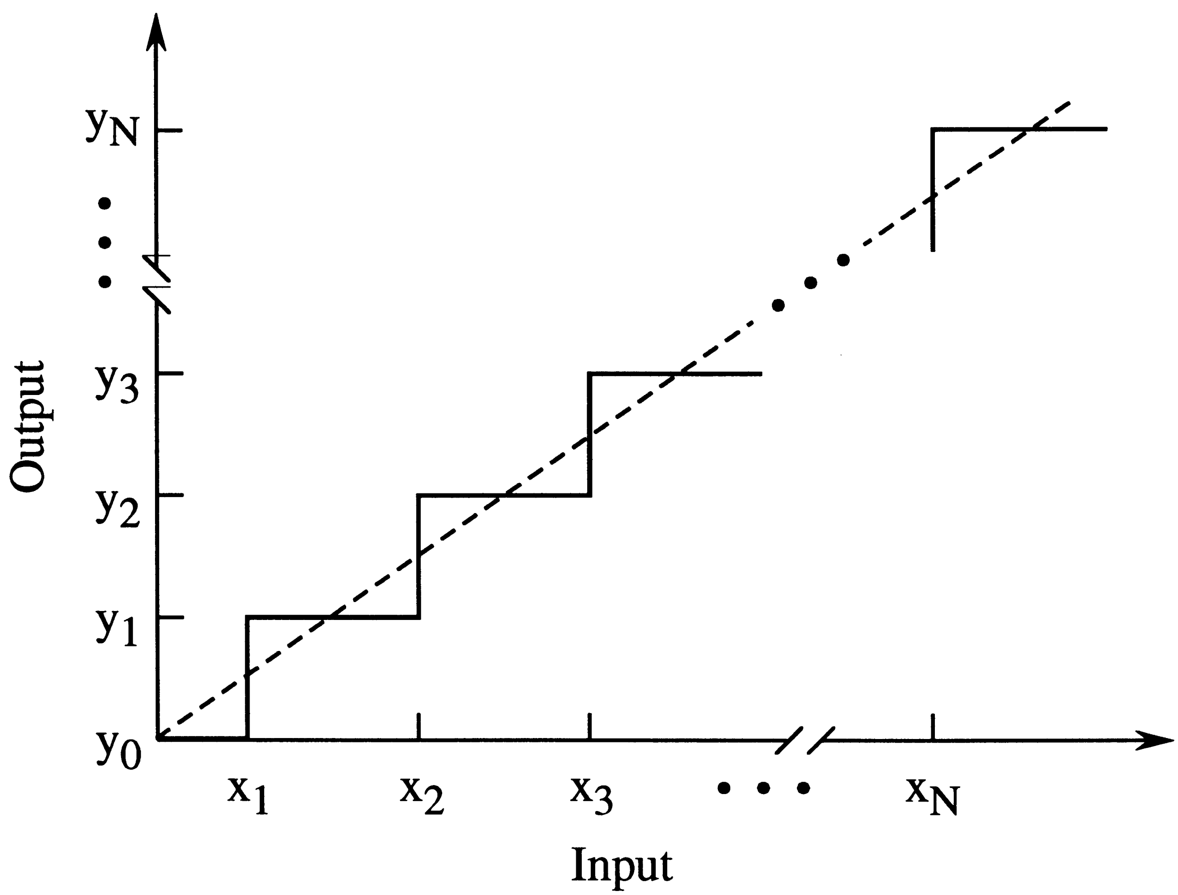
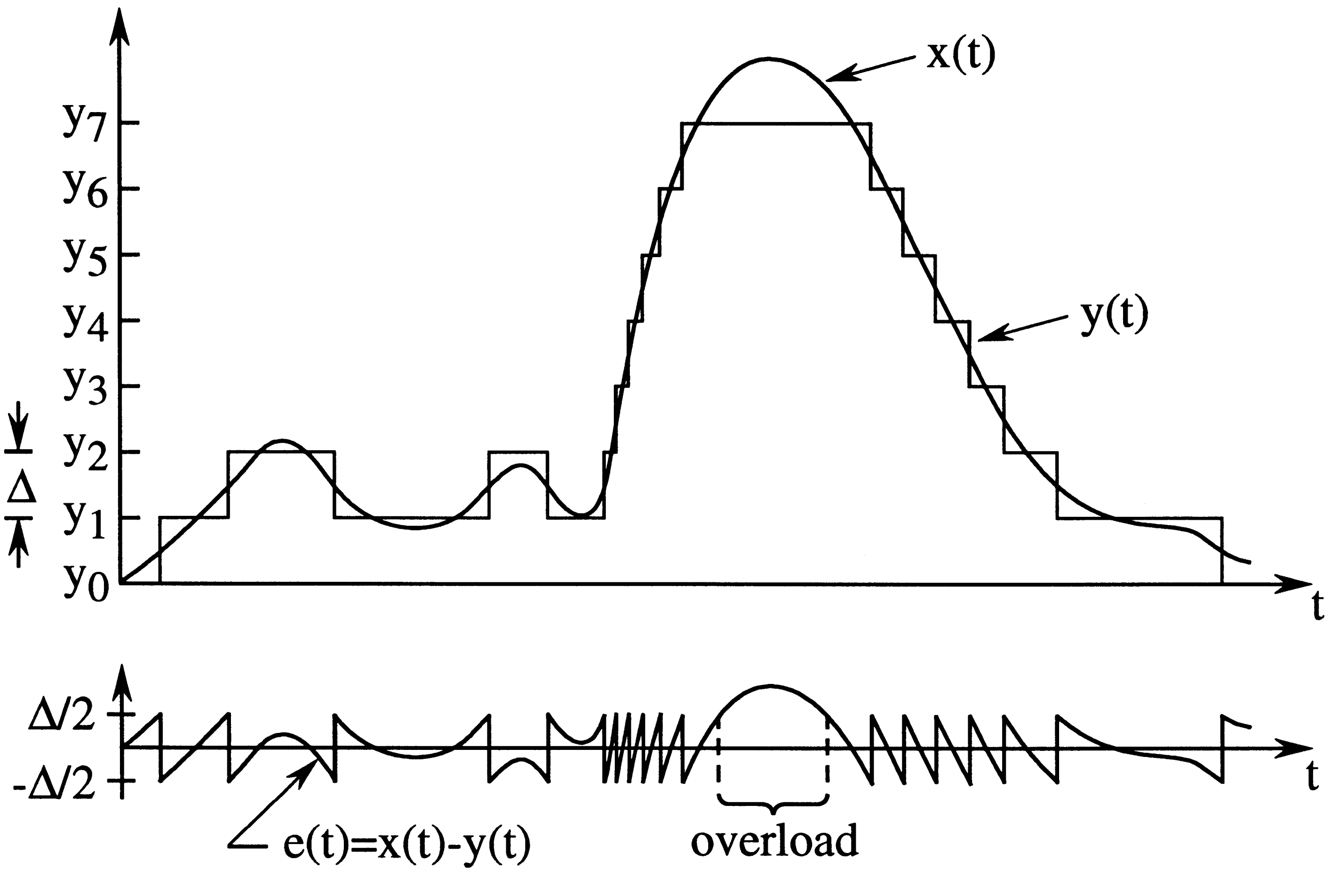
2.3.4 QUANTIZATION

- Quantization refers to the process whereby a continuum of amplitude values is represented by a finite set of discrete values



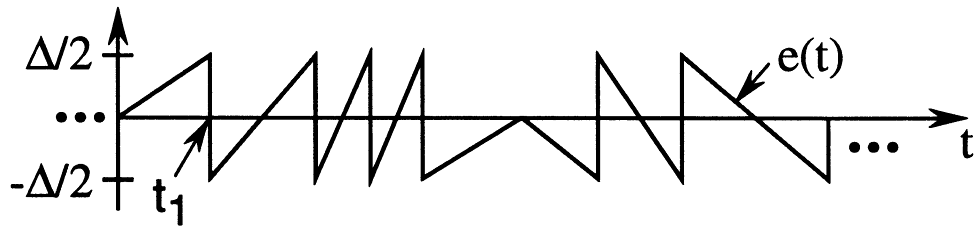
$$Q(x) = y_k,\; x_k < x \leq x_{k+1}$$

Quantization of a Waveform



Quantization Error Statistics

Deterministic Sawtooth Waveform Error Model



Mean-Squared Value

$$e_{ms} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |e(t)|^2 dt$$

$$= \frac{1}{t_1} \int_{0}^{t_1} \left| \frac{\Delta/2}{t_1} t \right|^2 dt = \frac{1}{t_1^3} \frac{\Delta^2}{4} \frac{t^3}{3} \Bigg|_{0}^{t_1}$$

$$= \frac{\Delta^2}{12}$$

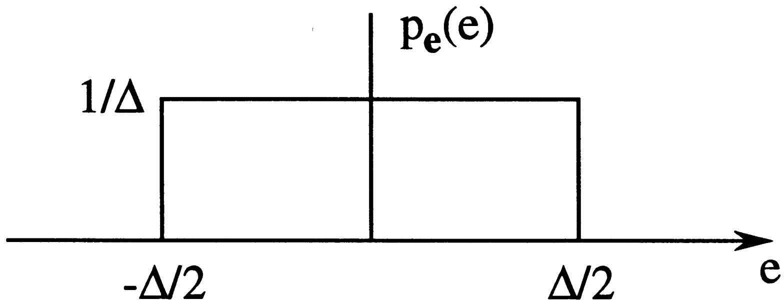
Mean Value

$$e_{avg} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} e(t)dt$$

$$= 0 \text{ by symmetry}$$

Stochastic Model

Let **e**(t) be a random variable uniformly distributed on the interval $\left[-\frac{\Delta}{2}, \frac{\Delta}{2}\right]$



Probability density function for the quantization error

Mean Value

$$E\{\mathbf{e}(t)\} = \int e\, p_{\mathbf{e}}(e)de$$

$$= 0 \text{ by symmetry}$$

Mean-Squared Value

$$E\{|\mathbf{e}(t)|^2\} = \int e^2 p_{\mathbf{e}}(e)de$$

$$= \frac{1}{\Delta} \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} e^2 de = \frac{1}{\Delta} \frac{e^3}{3} \Bigg|_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}}$$

$$= \frac{\Delta^2}{12}$$

Signal-to-Noise Ratio for Uniform Quantizer

Let $\mathbf{x}(t)$ be uniformly distributed on interval $[-X, X)$. Assume a B bit quantizer

$$N = 2^B$$

$$\Delta = \frac{2X}{N} = 2^{-(B-1)}X$$

Signal power $\quad E\{|\mathbf{x}(t)|^2\} = \frac{4X^2}{12}$

Noise power $\quad E\{|\mathbf{e}(t)|^2\} = \frac{\Delta^2}{12} = 2^{-2(B-1)}X^2$

Signal-to-Noise Ratio

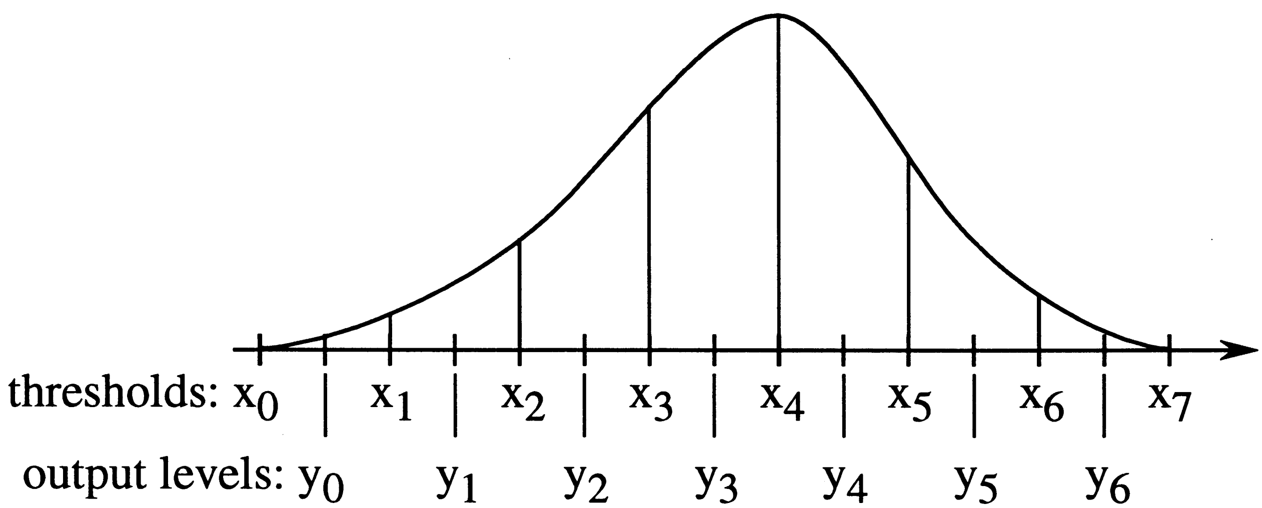
$$\mathrm{SNR} = \frac{X^2/3}{2^{-2(B-1)}X^2} = \frac{1}{12}\,2^{2B}$$

$$= 6B - 10.8 \ \ \mathrm{dB}$$

General form of expression remains the same for other densities $p_x(x)$; only the constant changes.

Optimum (Nonuniform) Quantizer

Let the signal **x**(t) be a random variable with probability density function $p_{\mathbf{x}}(x)$



How do we choose optimum threshold levels and output values?

Mean-Squared Quantizer Error

$$E\{|\mathbf{e}(t)|^2\} = E\{|\mathbf{x}(t) - \mathbf{y}(t)|^2\}$$

$$= E\{|\mathbf{x}(t) - Q[\mathbf{x}(t)]|^2\}$$

$$= \int |x - Q[x]|^2 p_{\mathbf{x}}(x)dx$$

$$= \sum_{k=0}^{N-1} \int_{x_k}^{x_{k+1}} [x - y_k]^2 p_{\mathbf{x}}(x)dx$$

$$E\{|\mathbf{e}(t)|^2\} = \sum_{k=0}^{N-1} \int_{x_k}^{x_{k+1}} [x - y_k]^2 p_{\mathbf{x}}(x)dx$$

For fixed ℓ, differentiate with respect to y_ℓ

$$\frac{\partial E\{|\mathbf{e}(t)|^2\}}{\partial y_\ell} = \int_{x_\ell}^{x_{\ell+1}} 2[x - y_\ell]p_{\mathbf{x}}(x)dx$$

Set derivative equal to zero

$$y_\ell = \frac{\int_{x_\ell}^{x_{\ell+1}} x\, p_{\mathbf{x}}(x)dx}{\int_{x_\ell}^{x_{\ell+1}} p_{\mathbf{x}}(x)dx} = E\{\mathbf{x} \,|\, x_\ell < \mathbf{x} \leq x_{\ell+1}\}$$

$$E\{|\mathbf{e}(t)|^2\} = \sum_{k=0}^{N-1} \int_{x_k}^{x_{k+1}} [x - y_k]^2 p_{\mathbf{x}}(x)dx$$

For fixed ℓ, differentiate with respect to x_ℓ

$$\frac{\partial E\{|\mathbf{e}(t)|^2\}}{\partial x_\ell} = [x_\ell - y_{\ell-1}]^2 p_{\mathbf{x}}(x_\ell) - [x_\ell - y_\ell]^2 p_{\mathbf{x}}(x_\ell)$$

Set derivative equal to zero

$$x_\ell = \frac{1}{2}[y_\ell + y_{\ell-1}]$$

Summarizing

$$y_\ell = \frac{\displaystyle\int_{x_\ell}^{x_{\ell+1}} x\, p_{\mathbf{x}}(x)dx}{\displaystyle\int_{x_\ell}^{x_{\ell+1}} p_{\mathbf{x}}(x)dx}, \quad \ell = 0,\ldots,N-1$$

$$x_\ell = \begin{cases} -\infty, & \ell = 0 \\ \frac{1}{2}[y_\ell + y_{\ell-1}], & \ell = 1,\ldots,N-1 \\ \infty, & \ell = N \end{cases}$$

Comments

- These equations must be solved iteratively.
- The two necessary conditions for optimality were independently reported by Lukaszewicz and Steinhaus (1955), Lloyd (1957), and Max (1960).
- These ideas can be generalized to the quantization of vector-valued signals.
  - color image quantization
  - image compression

Image Quantization

- The primary artifacts caused by image quantization are contouring and blockiness.

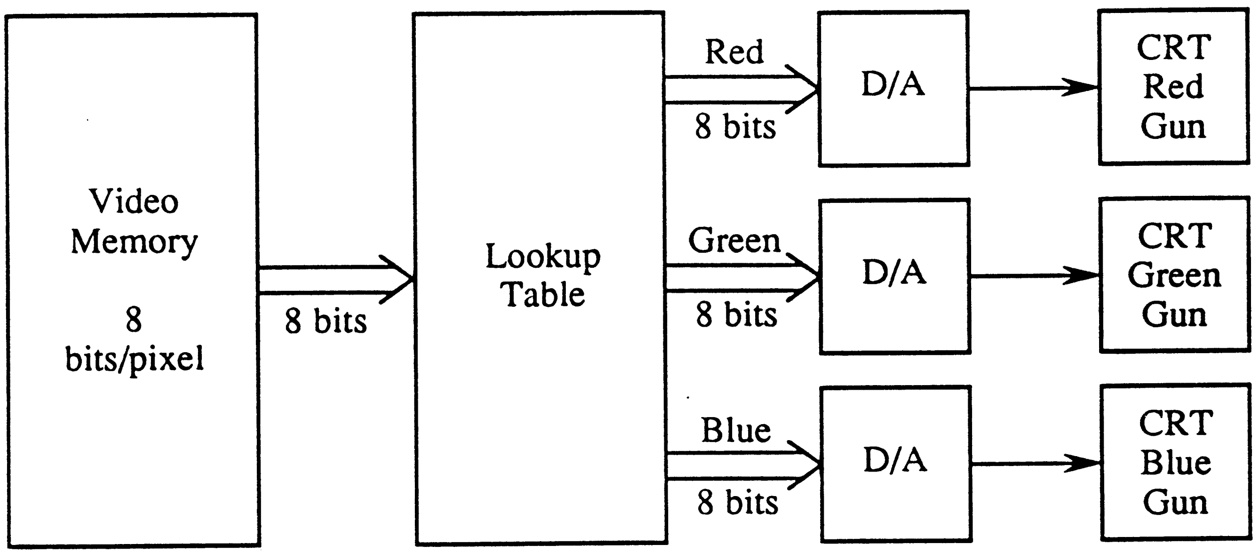
- Generally, 256 levels (8 bits) per pixel will be sufficient to prevent the appearance of such artifacts in monochrome images.

- If fewer levels must be used, halftoning techniques may be employed to increase the number of effective output levels at the possible expense of texturing or noise artifacts and some loss of detail.

COLOR IMAGE PALETTIZATION

- With 24 bits/pixel of video memory, color images may be displayed directly without artifacts.
- Many color displays have only 8 bits of video memory which are mapped into a lookup table (LUT) with 24 bits at the output.

TYPICAL DISPLAY ARCHITECTURE



- Choosing the best palette of 256 colors from the full set of 2^{24} possible colors is a quantization problem.
- Three steps in palettization:
    1. palette design which results in a set of 256 24-bit color vectors C_k, k = 0, 1, ..., 255.
    2. mapping every 24-bit pixel in the image to the 8 bit label k corresponding to an output color C_k from the palette.
    3. mapping every pixel in the label image to a 24-bit output color (accomplished via display hardware LUT).

TECHNIQUES FOR PALETTE DESIGN

IMAGE INDEPENDENT METHODS:

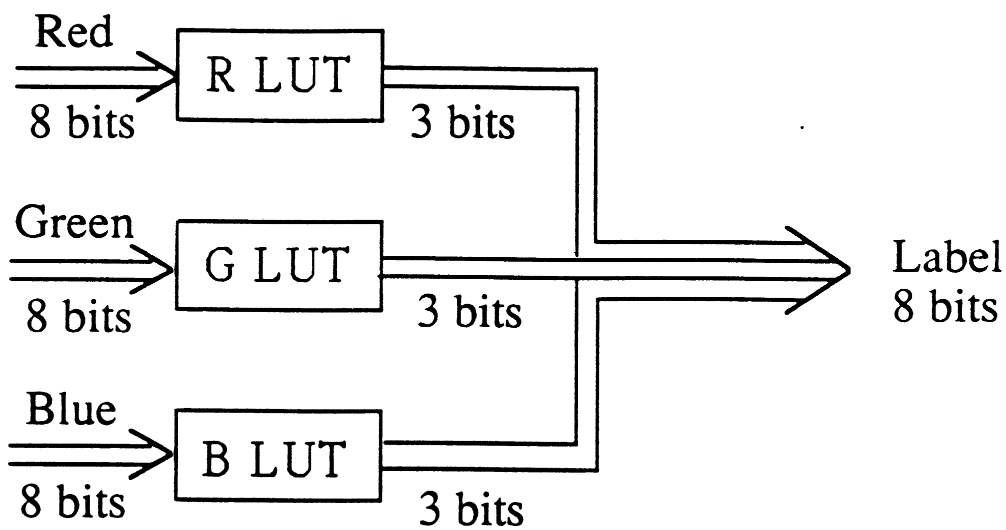
- **Scalar Quantization of R, G, and B** (Goertzel and Thompson, 1990)
  - Image is prequantized to 6 bits/color (0 – 63 in digital value).
  - Red and Green are quantized to 7 levels each, and Blue is quantized to 4 levels for a total of $7 \times 7 \times 4 = 196$ colors.

— The remaining 60 colors are reserved for other uses.

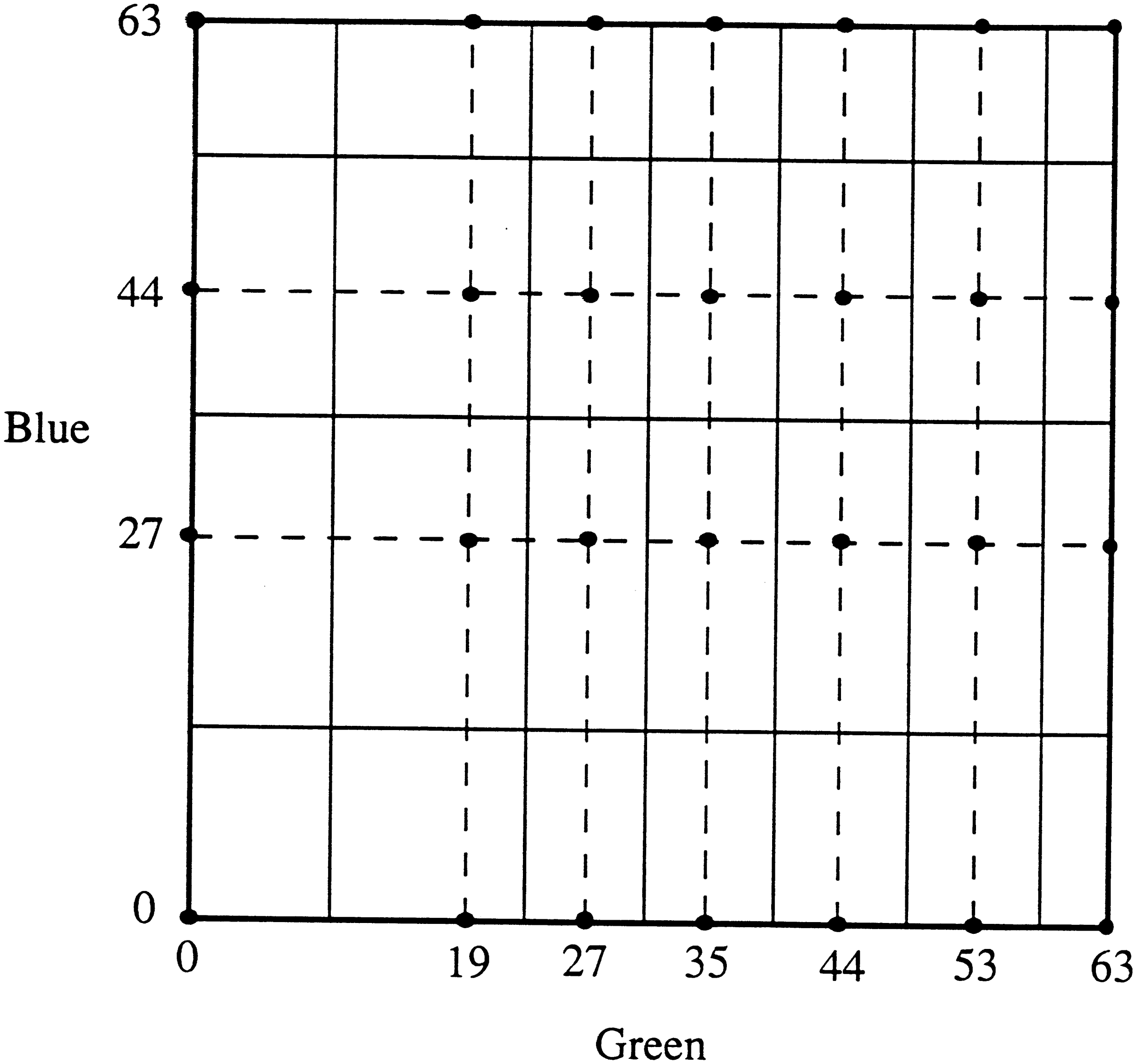
— The output colors are spaced nonuniformaly to yield equal steps in L^* when only one primary is nonzero.

— Halftoning is needed for reasonable quality.

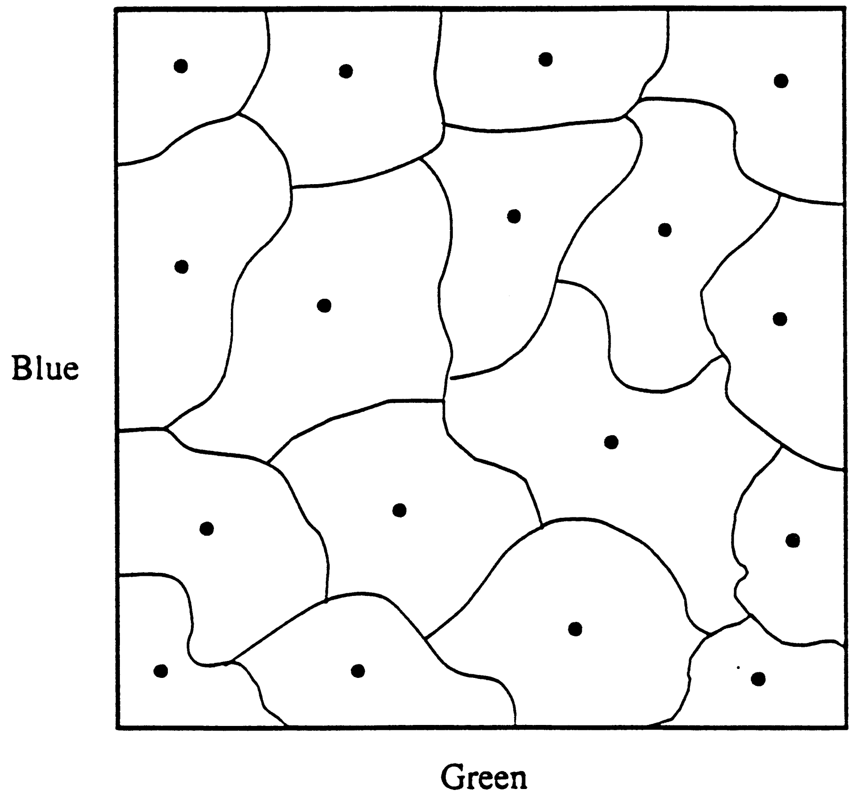
— Mapping can be done via 3 LUT's



SCALAR QUANTIZATION OF R, G, AND B



- **Application of image-dependent techniques to a uniform histogram** (Gentile, Allebach, and Walowit, 1990).



  — Since the palette has no structure, the mapping requires a search for the nearest output color.

  — Halftoning is still needed for reasonable quality.

IMAGE DEPENDENT METHODS:

- **Based on histogram of image**
- **Splitting Methods**

— **Splits orthogonal to coordinate axes**

Median cut (Heckbert, 1982)

Split along coordinate with greatest range at median point of that coordinate.

Variance-based method (Wan et al., 1990)

Split region with greatest total squared error (TSE) along coordinate that results in greatest reduction in TSE. Split region at centroid.

Splitting Orthogonal to Coordinate Axes

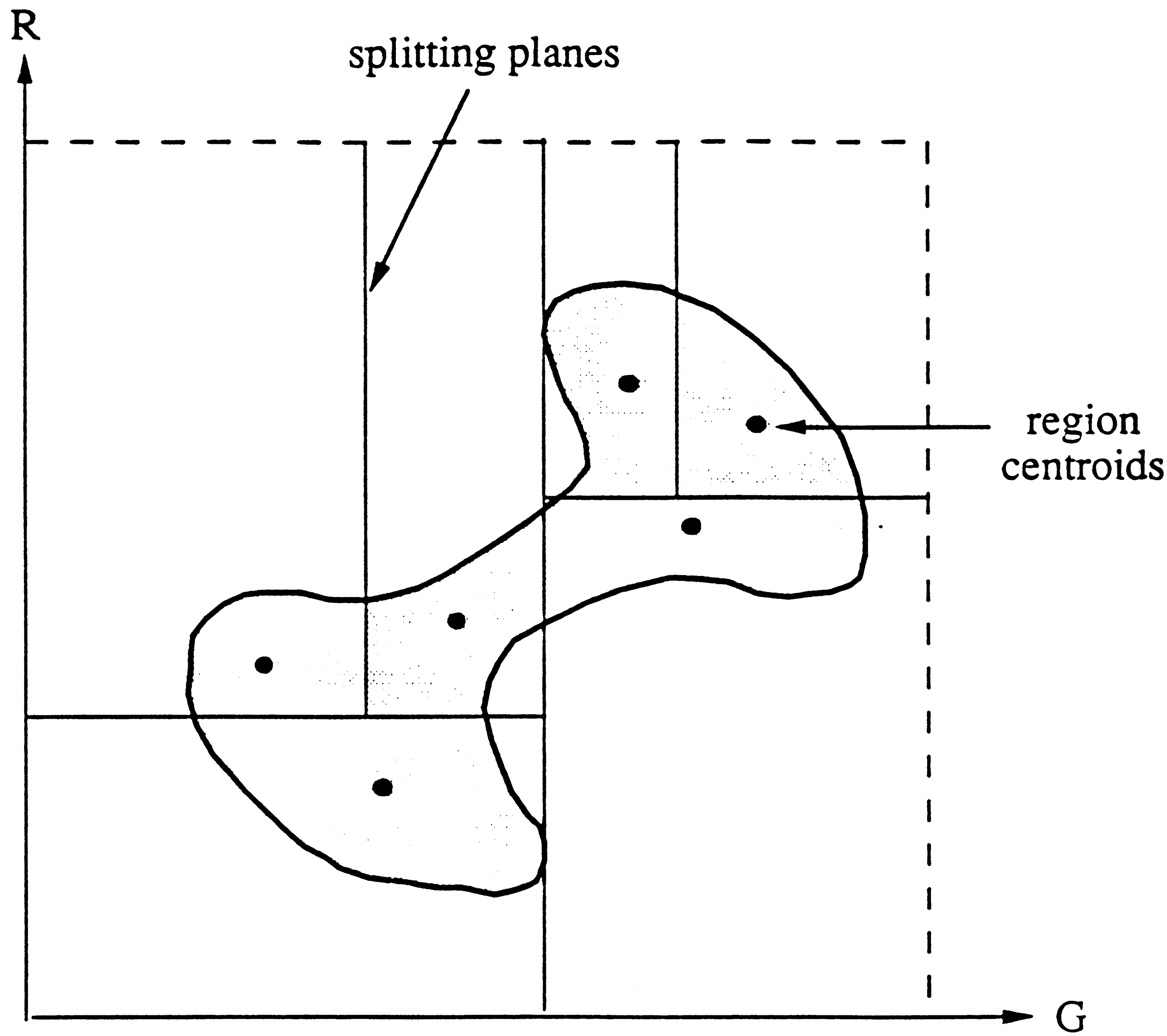


Figure 1: Splitting of Color Space by Median Cut and Variance Based Algorithms

— Splits orthogonal to direction of greatest variation (Orchard and Bouman, 1991)

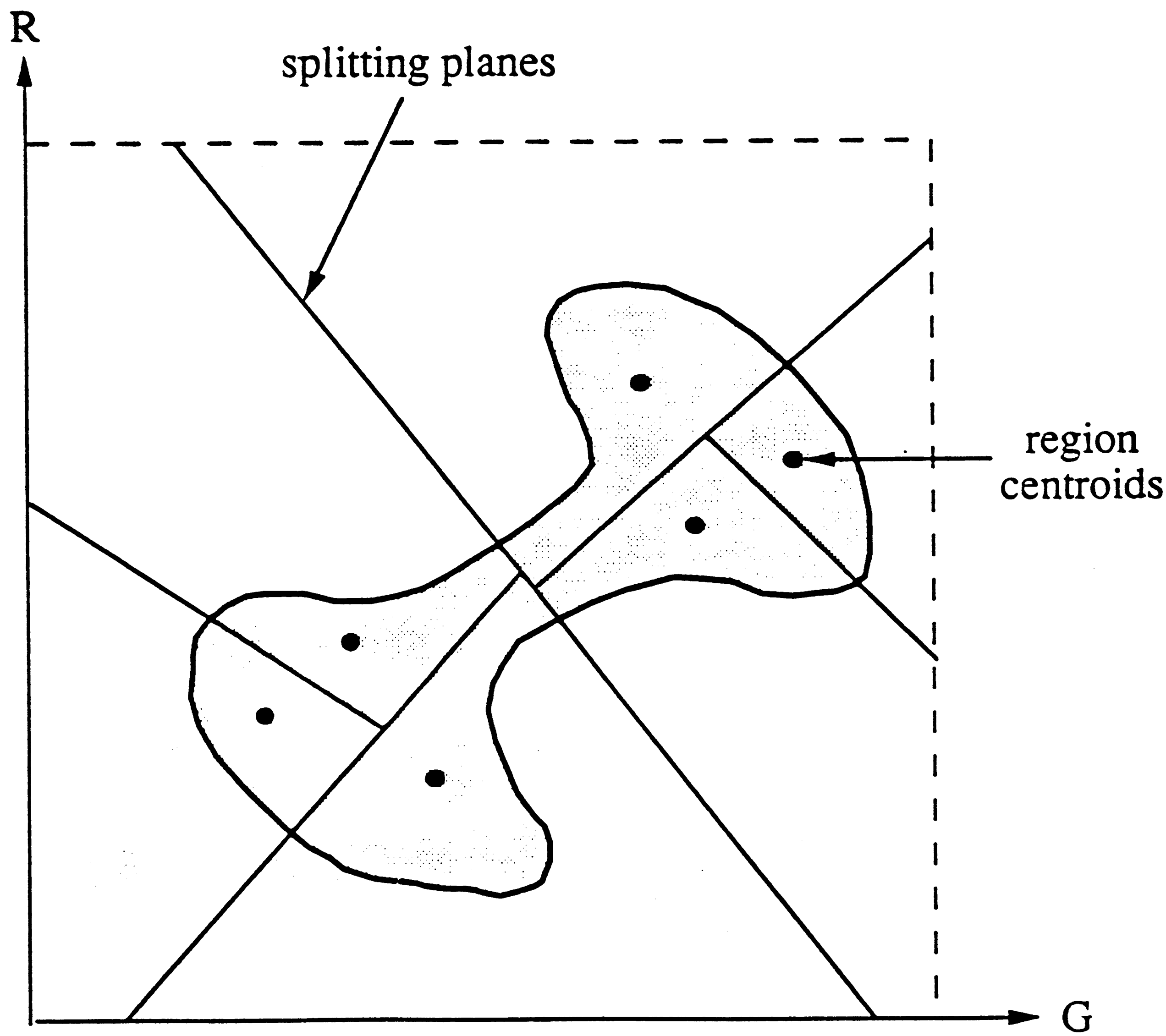


Figure 2: Splitting of Color Space by Binary Splitting Algorithm

- **Merging Methods** (Balasubramanian and Allebach, 1991)

  — Start with every image color assigned to a separate cluster

  — Apply Equitz's (1989) pairwise nearest neighbor merging technique to iteratively merge "nearest" clusters until desired number of clusters is obtained

  — K-D trees are used to efficiently organize search

Peak-Based Methods (Braudaway, 1987).

— Choose color coordinate corresponding to maximum value of histogram as an output color

— Reduce histogram in neighborhood of this color by applying a weighting that increases exponentially with distance from the chosen color.

— Repeat the above two steps until the desired number of colors is obtained.

Refinement Techniques (Linde, Buzo, and Gray, 1980)

— Given a set of quantization cells, choose new output color for each cell as the centroid of that cell.

— Given a set of output colors, determine new quantization cells by mapping each color to the nearest output color.

— Repeat the above steps until convergence.

— Method is computationally intensive.

— Applied to color image quantization by Heckbert (1982), Braudaway (1987), and Gentile, Allebach, and Walowit (1990).

- **Incorporation of Spatial Activity Measures** (Orchard and Bouman, 1991) (Balasubramanian and Allebach, 1991)

  — The human viewer is less sensitive to quantization errors in "busy" areas of the image that contain significant spatial detail.

  — Weight distance in the color space inversely with the spatial activity in regions of the image where the colors occur.

  — Divide image into 8×8 blocks P_k

$$\alpha_k = \frac{1}{64} \sum_{p \epsilon P_k} \| C_p - \overline{C_k} \|_1$$

  — Assign each color C an activity measure

$$\tilde{\alpha}_C = \min_{k \epsilon K_C} \alpha_k$$

- **Prequantization and Efficient Histogramming** (Balasubramanian and Allebach, 1991)

  — Reduce the number of distinct colors in the image by prequantization. Can also be done in a manner that accounts for spatial activity.

  — Use digital values for R and G to index into a 2-D array. Each location in the array is the root of a tree or linked list. The nodes of the tree or list contain the B values of colors that occur in the image with the particular RG value and the number of times that the color occurs.

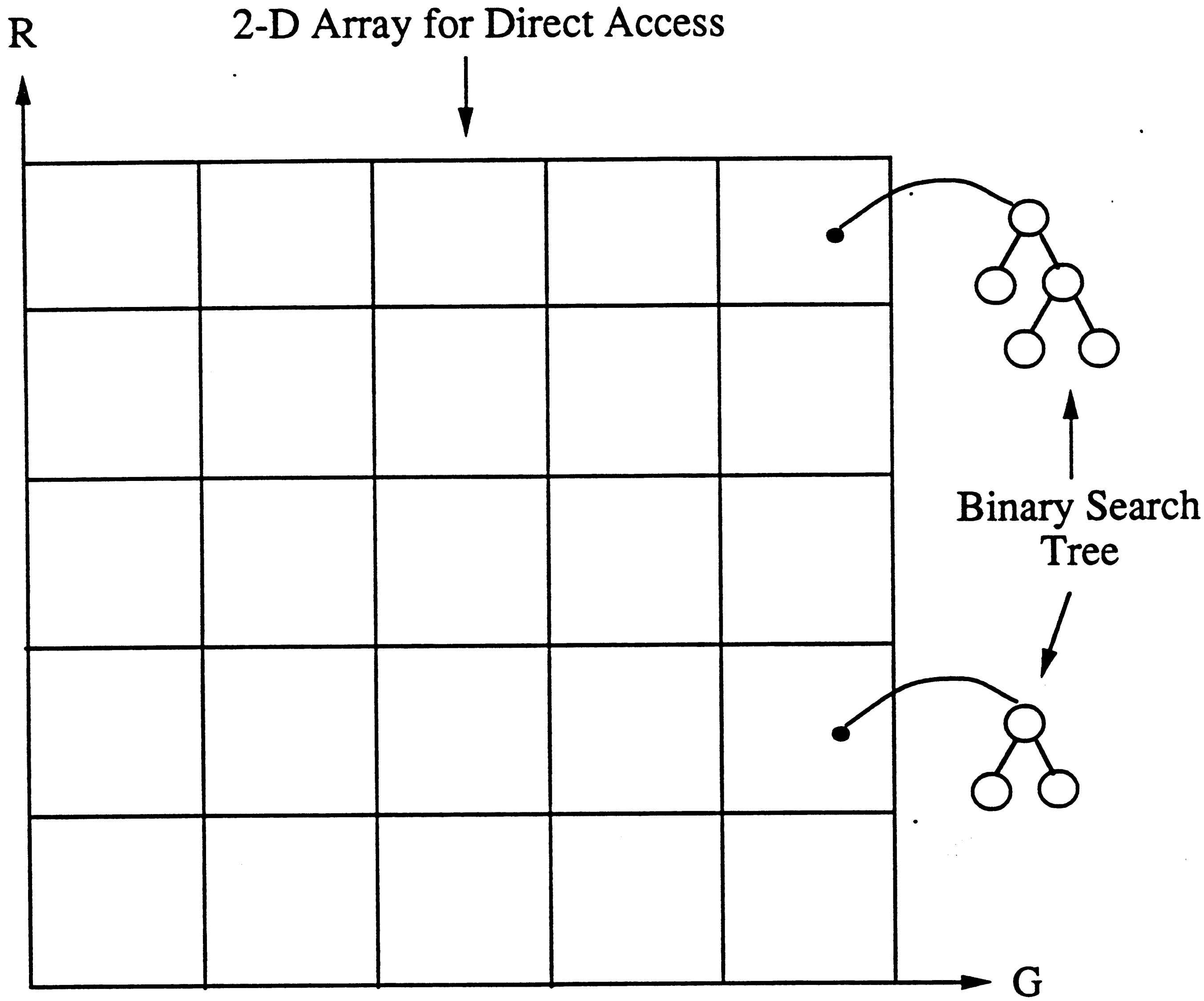


Figure 3: Color Histogram with Direct Access for Sorting by **R** and **G** Values and Binary Search Trees for Sorting by **B** Value

- **Mapping the Image Color to the Palette**

— The palette designed by splitting techniques has a tree structure that allows for efficient mapping.

— The palette designed by other methods has no structure. Mapping can be accomplished by nearest neighbor search or by keeping track of the cluster to which each pixel belongs throughout the palette design.

• Computational Complexity

N_p - image size

N_c - number of distinct colors in the image

M - palette size

— preprocessing (prequantization, activity measure, and histogram) $O(N_p)$

— splitting $O(N_c \log_2 M)$

— mapping $O(N_p \log_2 M)$

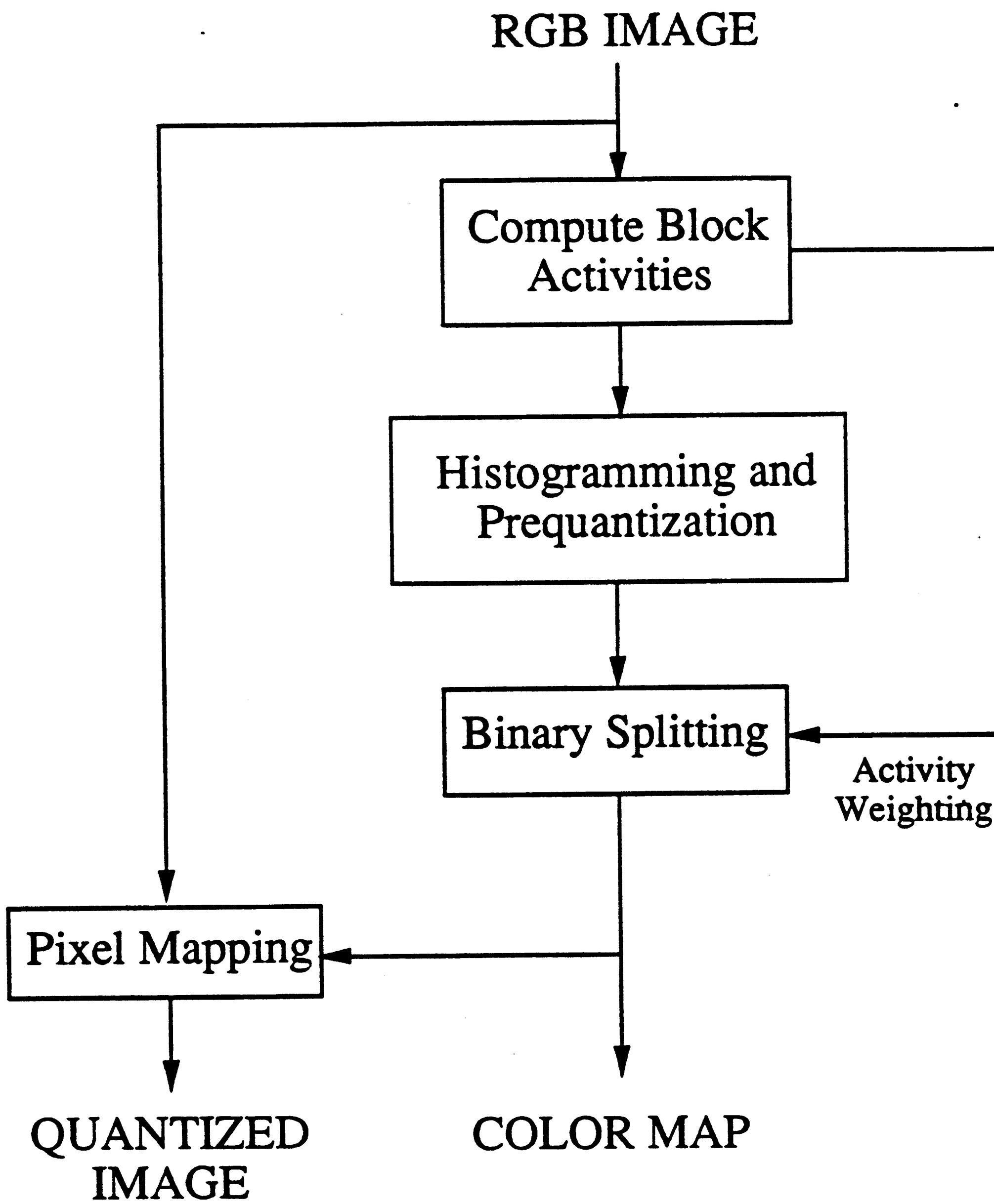


Figure 5: Summary of Quantization Algorithm