Digital Filter Design

Synopsis

- Overview of filter design problem
- Finite impulse response filter design
- Infinite impulse response filter design

IIR vs. FIR Filters

- **FIR filter equation**

$$y[n] = a_0 x[n] + a_1 x[n-1] + \cdots + a_{M-1} x[n-(M-1)]$$

- **IIR filter equation**

$$y[n] = a_0 x[n] + a_1 x[n-1] + \cdots + a_{M-1} x[n-(M-1)]$$
$$- b_1 y[n-1] - b_2 y[n-2] - \cdots - b_N y[n-N]$$

- **IIR filters can generally achieve given desired response with less computation than FIR filters**
- **It is easier to approximate arbitrary frequency response characteristics with FIR filters, including exactly linear phase**

Infinite Impulse Response (IIR) Filter Design: Overview

- IIR digital filter designs are based on established methods for designing analog filters
- Approach is generally limited to frequency selective filters with ideal passband/stopband characteristics
- Basic filter type is low pass
- Achieve highpass or bandpass via transformations
- Achieve multiple stop/pass bands by combining multiple filters with single pass band

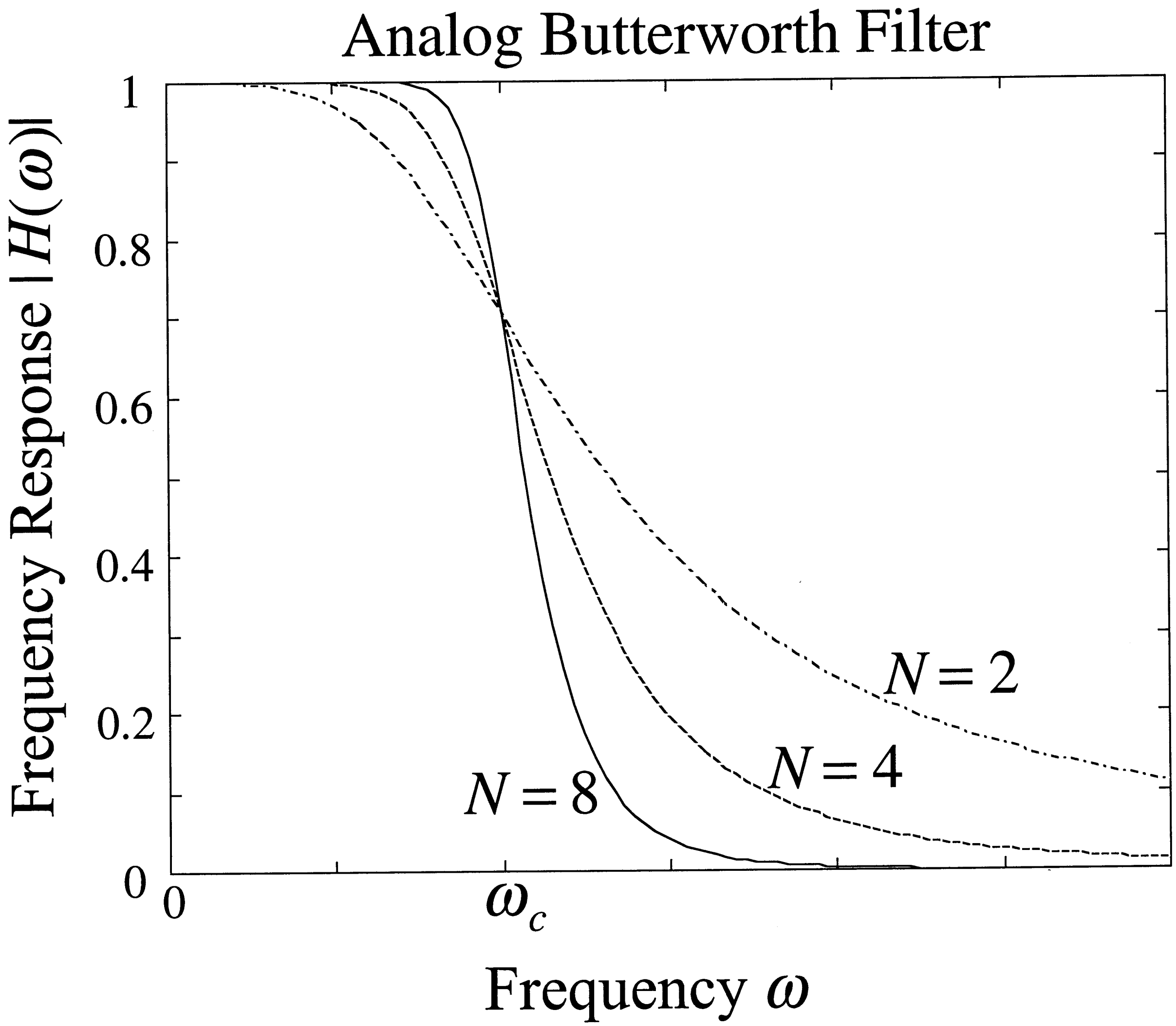
IIR Filter Design Steps

- Choose prototype analog filter family
  - Butterworth
  - Chebychev Type I or II
  - Elliptic
- Choose analog-digital transformation method
  - Impulse Invariance
  - Bilinear Transformation

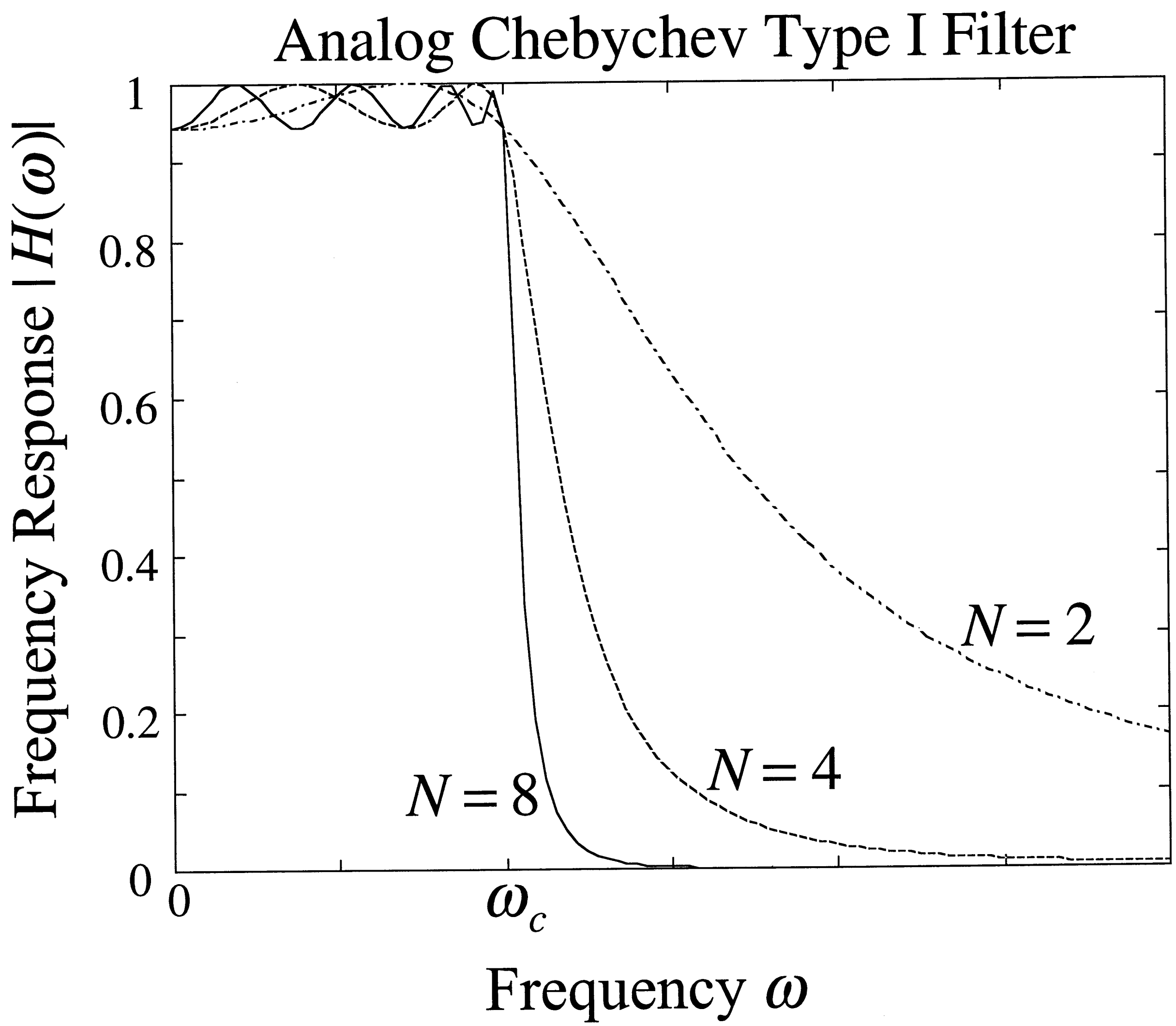
IIR Filter Design Steps (cont.)

- **Transform digital filter specifications to equivalent analog filter specifications**
- **Design analog filter**
- **Transform analog filter to digital filter**
- **Perform frequency transformation to achieve highpass or bandpass filter, if desired**

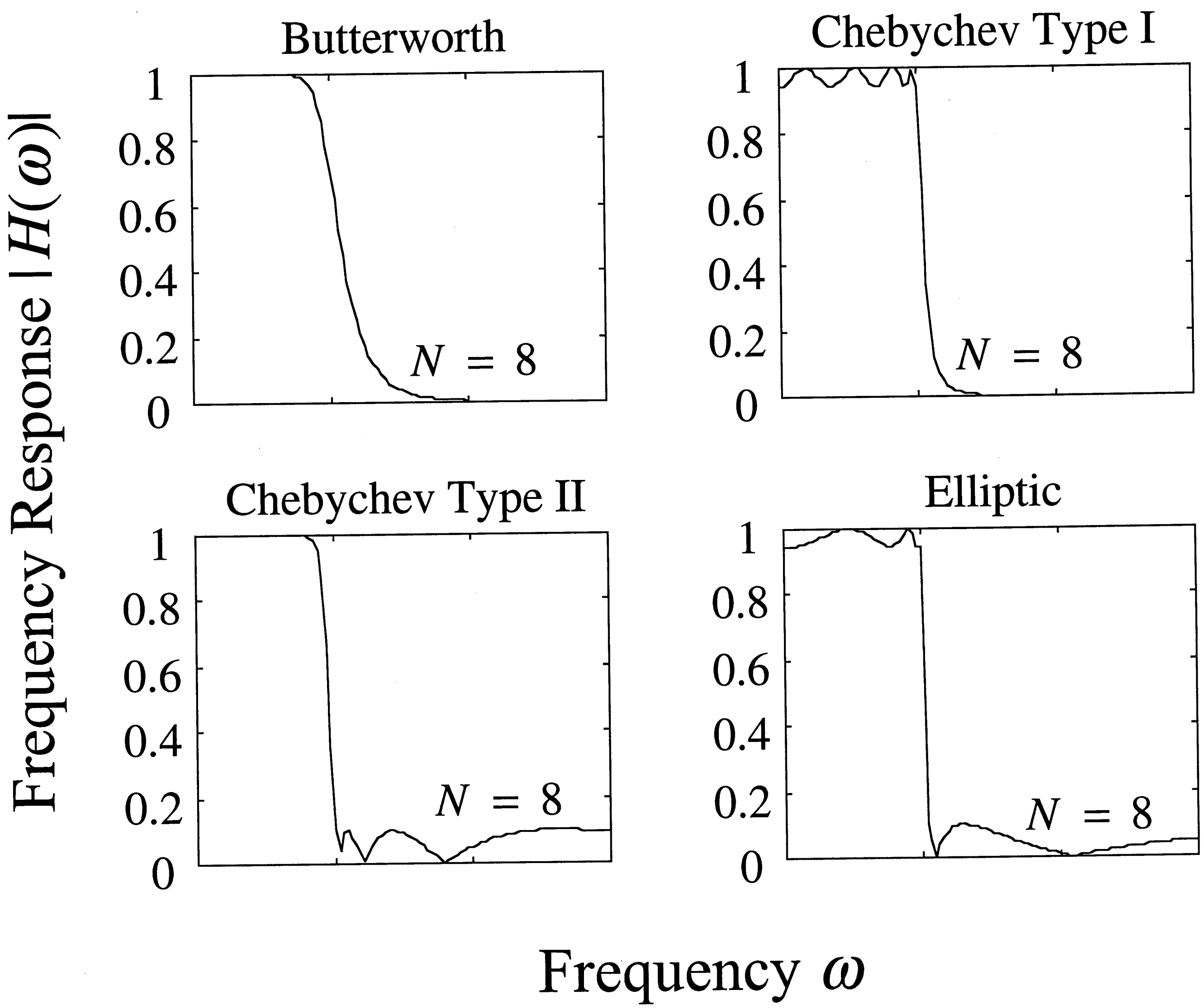
Prototype Filter Types



Prototype Filter Types (cont.)



Prototype Filter Types (cont.)



Transformation via Impulse Invariance

- **Sample impulse response of analog filter**

$$h_{\mathrm{d}}[n] = T\,h_{\mathrm{a}}(nT)$$

$$H_{\mathrm{d}}(\omega) = \sum_k H_{\mathrm{a}}(f - k/T)\Big|_{f=\frac{\omega}{2\pi T}}$$

  - **Note that aliasing may occur**

Impulse Invariance (cont.)

- **Implementation of digital filter**
  - **Partial fraction expansion of analog transfer function (assuming all poles have multiplicity 1)**

$$H_a(s) = \sum_{k=1}^{N} \frac{A_k}{s - s_k}$$

  - **Inverse Laplace transform**

$$h_a(t) = \sum_{k=1}^{N} A_k e^{s_k t} u(t)$$

Impulse Invariance (cont.)

– **Sample impulse response**

$$h_{\mathrm{d}}[n] = Th_{\mathrm{a}}(nT) = T\sum_{k=1}^{N} A_k e^{s_k nT} u(nT)$$

$$= T\sum_{k=1}^{N} A_k\, p_k^n\, u[n], \quad p_k = e^{s_k T}$$

– **Take Z transform**

$$H_{\mathrm{d}}(z) = \sum_{k=1}^{N} \frac{TA_k}{1 - p_k z^{-1}}$$

Impulse Invariance (cont.)

– **Combine terms**

$$H_{\mathrm{d}}(z) = \frac{a_0 + a_1 z^{-1} + \cdots + a_{M-1} z^{-(M-1)}}{1 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_N z^{-N}}$$

– **Corresponding filter equation**

$$y[n] = a_0 x[n] + a_1 x[n-1] + \cdots + a_{M-1} x[n-(M-1)]$$

$$- b_1 y[n-1] - b_2 y[n-2] - \cdots - b_N y[n-N]$$

Impulse Invariance Example

- **Design a second order ideal low pass digital filter with cutoff at** $\omega_d = \pi / 5$ radians / sample

  (Assume $T = 1$)

  – **Analog cutoff frequency**

$$f_c = \frac{\omega_d}{2\pi T} = 0.1 \text{ Hz}$$

$$\omega_c = 2\pi f_c = \pi / 5 \text{ rad / sec}$$

- **Use Butterworth analog prototype**

$$H_a(s) = \frac{0.3948}{[s-(-0.4443 + j0.4443)][s-(-0.4443 - j0.4443)]}$$

Impulse Invariance Example (cont.)

- **Apply partial fraction expansion**

$$H_a(s) = \frac{-j0.4443}{[s-(-0.4443+j0.4443)]} + \frac{j0.4443}{[s-(-0.4443-j0.4443)]}$$

- **Compute inverse Laplace transform**

$$h_a(t) = \left[-j0.4443\,e^{(-0.4443+j0.4443)t} + j0.4443\,e^{(-0.4443-j0.4443)t}\right]u(t)$$

- **Sample impulse response**

$$h_d[n] = \left[-j0.4443\,e^{(-0.4443+j0.4443)n} + j0.4443\,e^{(-0.4443-j0.4443)n}\right]u[n]$$

$$= \left[-j0.4443\left(0.6413e^{j0.4443}\right)^n + j0.4443\left(0.6413e^{-j0.4443}\right)^n\right]u[n]$$

Impulse Invariance Example (cont.)

- **Compute Z Transform**

$$H_d(z) = \frac{-j0.4443}{1-0.6413e^{j0.4443}z^{-1}} + \frac{j0.4443}{1-0.6413e^{-j0.4443}z^{-1}}$$

- **Combine terms**

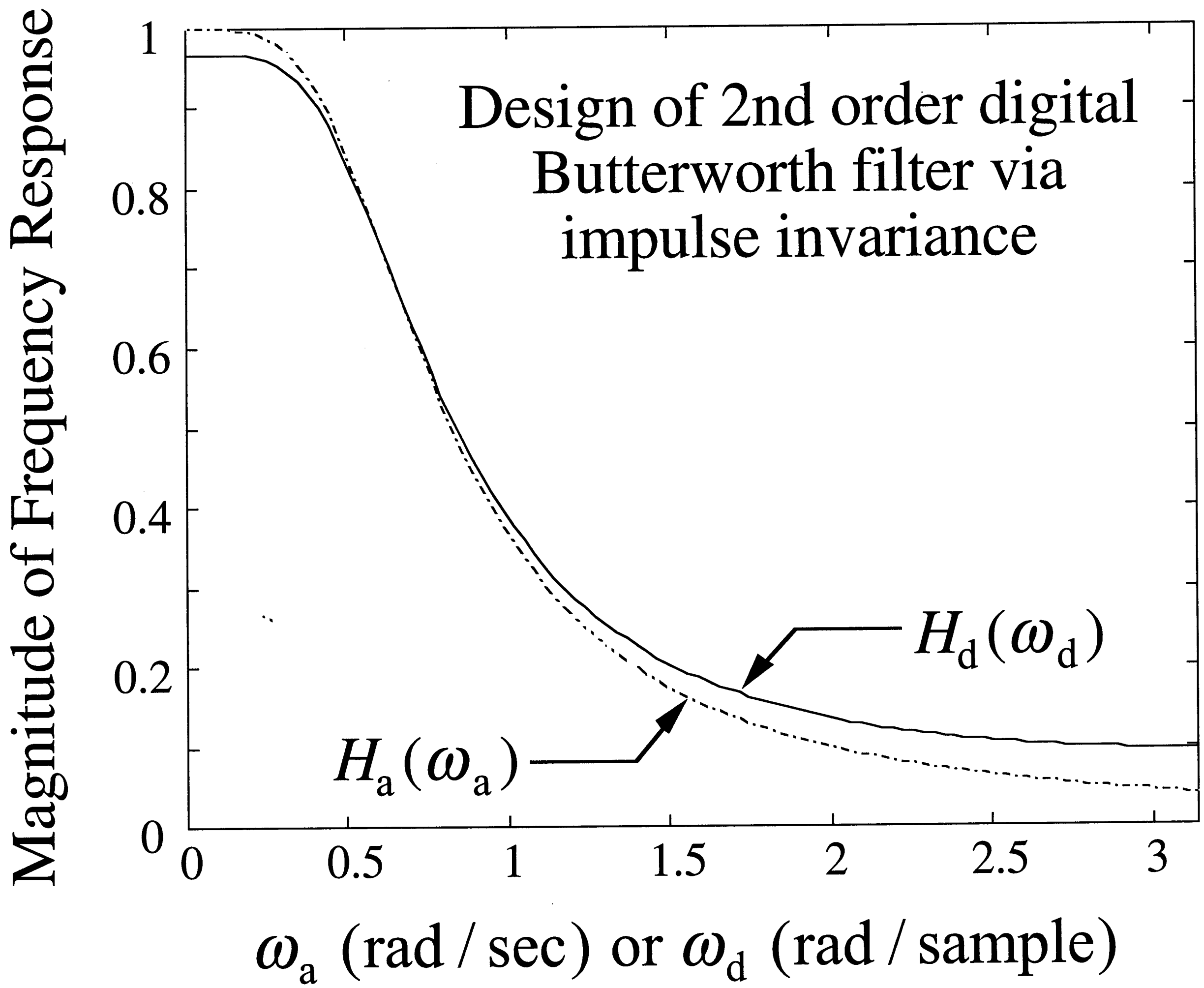
$$H_d(z) = \frac{0.2450z^{-1}}{\left(1-0.6413e^{j0.4443}z^{-1}\right)\left(1-0.6413e^{-j0.4443}z^{-1}\right)}$$

$$= \frac{0.2450z^{-1}}{1-1.1581z^{-1}+0.4113z^{-2}}$$

- **Find difference equation**

$$y[n] = 0.2450x[n-1] + 1.1581y[n-1] - 0.4113y[n-2]$$

Impulse Invariance Example (cont.)



Impulse Invariance Method - Summary

- **Preserves impulse response and shape of frequency response, if there is no aliasing**
- **Desired transition bandwidths map directly between digital and analog frequency domains**
- **Passband and stopband ripple specifications are identical for both digital and analog filters, assuming that there is no aliasing**
- **The final digital filter design is independent of the sampling interval parameter T**

Summary (cont.)

- Poles in analog filter map directly to poles in digital filter via transformation $p_k = e^{s_k T}$
- There is no such relation between the zeros in the two filters
- Gain at DC in digital filter may not equal unity, since sampled impulse response may only approximately sum to 1

Bilinear Transformation Method

- **Mapping between *s* and *z***

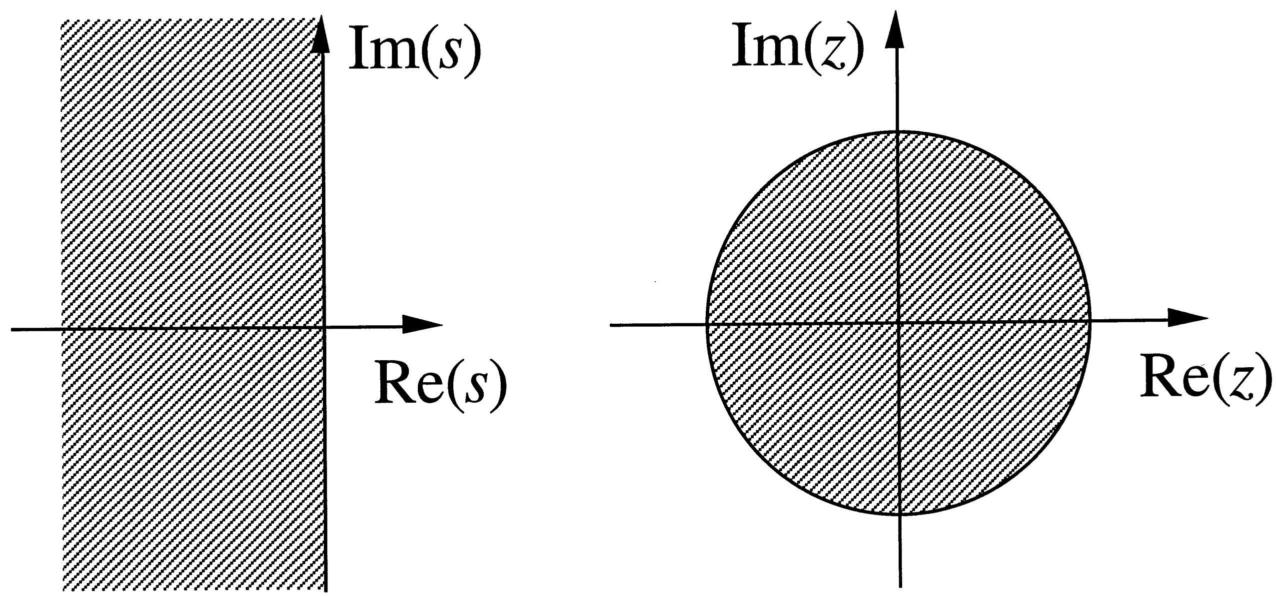
$$s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

$$H_{\mathrm{d}}(z) = H_{\mathrm{a}}\left[\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right]$$

$$z = \frac{1+(T/2)s}{1-(T/2)s}$$

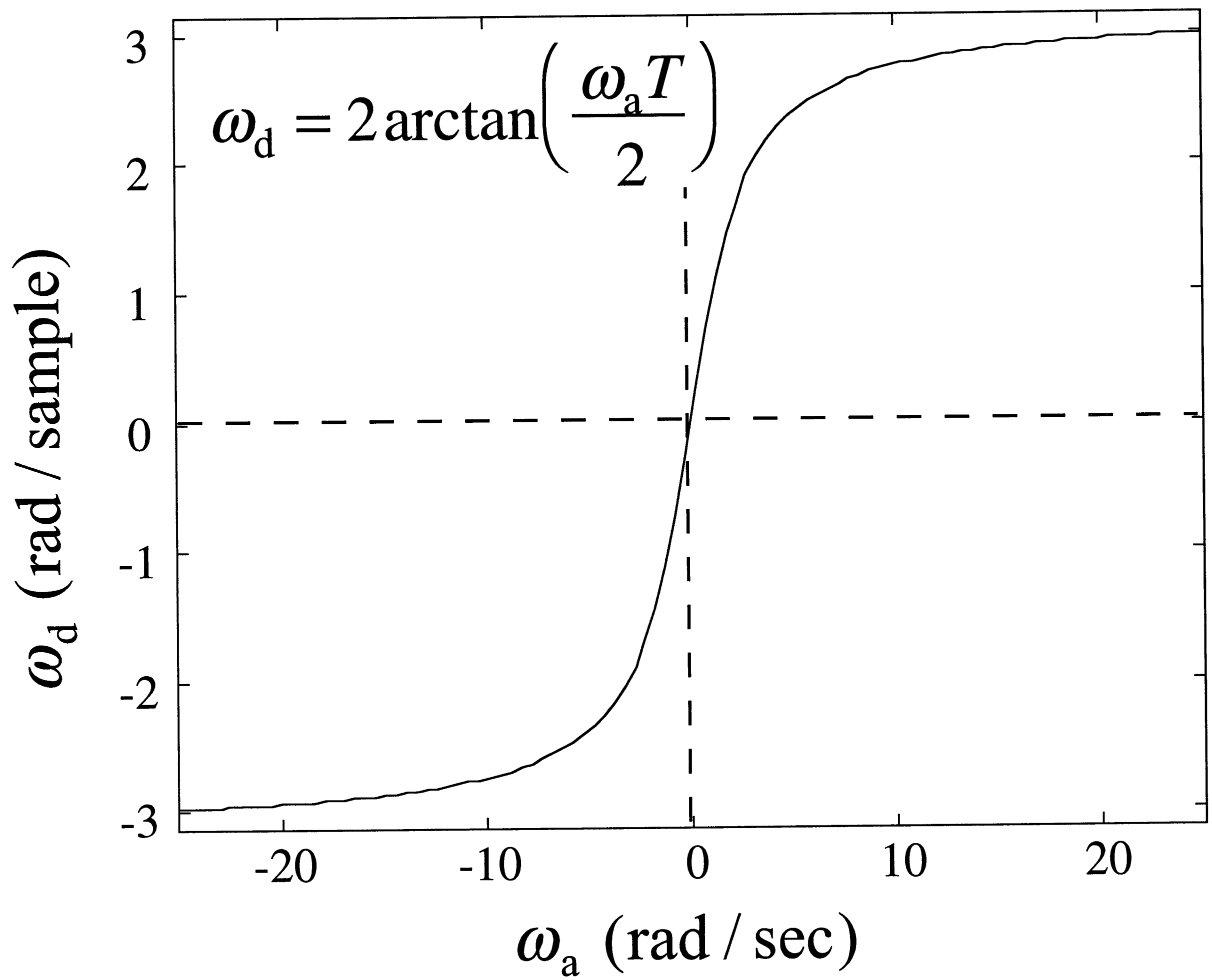
Bilinear Transformation (cont.)

- **Mapping between *s* and *z* planes**



Bilinear Transformation (cont.)

- **Mapping between analog and digital frequencies**



Bilinear Transformation - Example No. 1

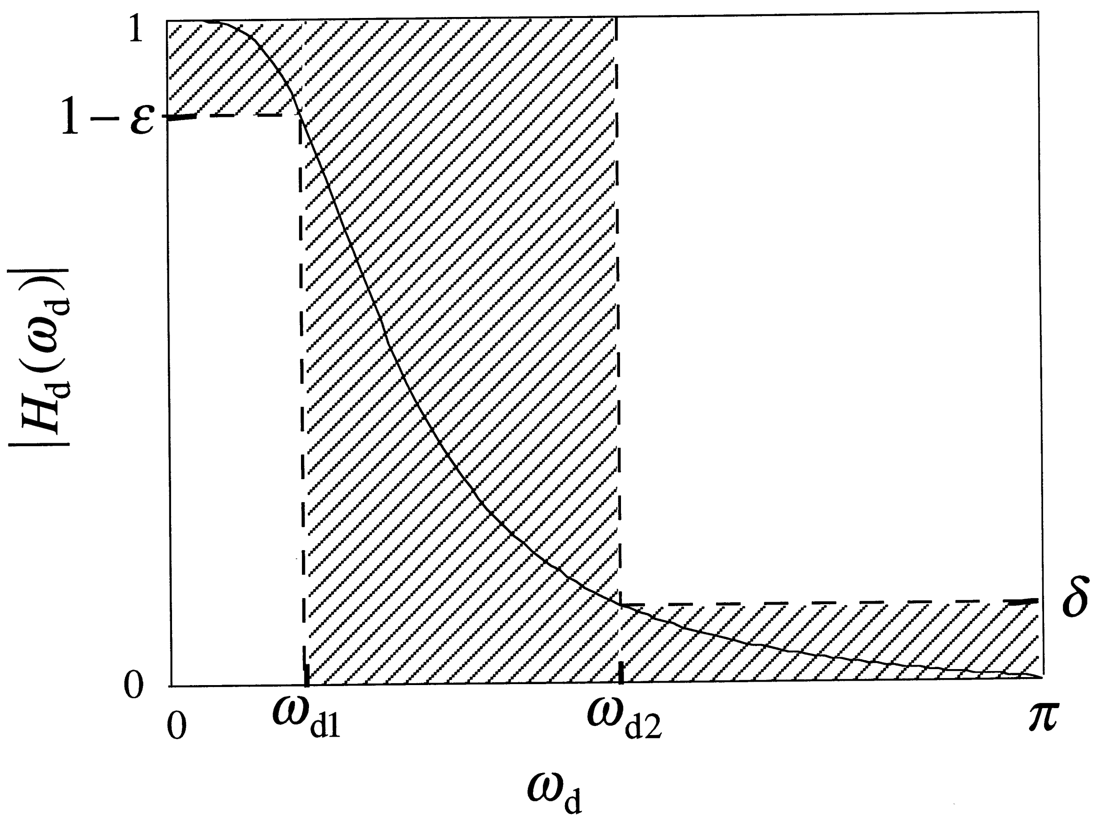
- **Design low-pass filter**
  - **cutoff frequency** $\omega_{dc} = \pi / 5 \text{ rad / sample}$
  - **transition bandwidth and ripple**

$$\begin{aligned}\Delta\omega_d &= 0.2\pi \text{ radians / sample} \\ &= 0.1 \text{ cycles / sample}\end{aligned}$$

$$\delta = 0.1$$

  - **use Butterworth analog prototype**
  - **set $T = 1$**

Bilinear Example No. 1 (cont.)



$$|H_d(\omega_{d1})| = 1 - \varepsilon$$

$$|H_d(\omega_{d2})| = \delta$$

Bilinear Example No. 1 (cont.)

- **Map digital to analog frequencies**

$$\omega_{\mathrm{a}} = \frac{2}{T}\tan(\omega_{\mathrm{d}}/2)$$

$$\omega_{\mathrm{d1}} = \pi/5 - 0.1\pi$$
$$\omega_{\mathrm{a1}} = 0.3168 \ \ \mathrm{rad/sec}$$

$$\omega_{\mathrm{d2}} = \pi/5 + 0.1\pi$$
$$\omega_{\mathrm{a2}} = 1.0191 \ \ \mathrm{rad/sec}$$

Bilinear Example No. 1 (cont.)

- **Solve for filter order and cutoff frequency**

$$|H_a(\omega_a)|^2 = \frac{1}{1+(\omega_a/\omega_c)^{2N}}$$

$$|H_a(\omega_{a1})| = 1-\varepsilon$$

$$|H_a(\omega_{a2})| = \delta$$

$$N = \left\lceil \frac{1}{2} \frac{\log\left[\left(|H_a(\omega_{a2})|^{-2}-1\right) / \left(|H_a(\omega_{a1})|^{-2}-1\right)\right]}{\log(\omega_{a2}/\omega_{a1})} \right\rceil$$

$$\omega_{ac} = \frac{\omega_{a2}}{\left(|H_a(\omega_{a2})|^{-2}-1\right)^{1/(2N)}}$$

Bilinear Example No. 1 (cont.)

- **Result:**

$$N = 3$$
$$\omega_{\text{ac}} = 0.4738$$

- **Determine transfer function of analog filter**

$$H_{\text{a}}(s)H_{\text{a}}(-s) = \frac{1}{1 + (s / j\omega_{\text{ac}})^{2N}}$$

- **Poles are given by**

$$s_k = \omega_{\text{ac}} e^{j\left(\frac{\pi}{2} + \frac{\pi}{2N} + \frac{2\pi k}{2N}\right)}, \quad k = 0, 1, \ldots, 2N - 1$$

Bilinear Example No. 1 (cont.)

- **Take N poles with negative real parts for *H(s)***

$$s_0 = -0.2369 + j0.4103 = 0.4738e^{j2\pi/3}$$
$$s_1 = -0.4738 + j0.0000 = 0.4738e^{j\pi}$$
$$s_2 = -0.2369 - j0.4103 = 0.4738e^{-j2\pi/3}$$

- **Transfer function of analog filter**

$$H_{\mathrm{a}}(s) = \frac{\omega_{\mathrm{ac}}^3}{(s-s_0)(s-s_1)(s-s_2)}$$

- **Transform to discrete-time filter**

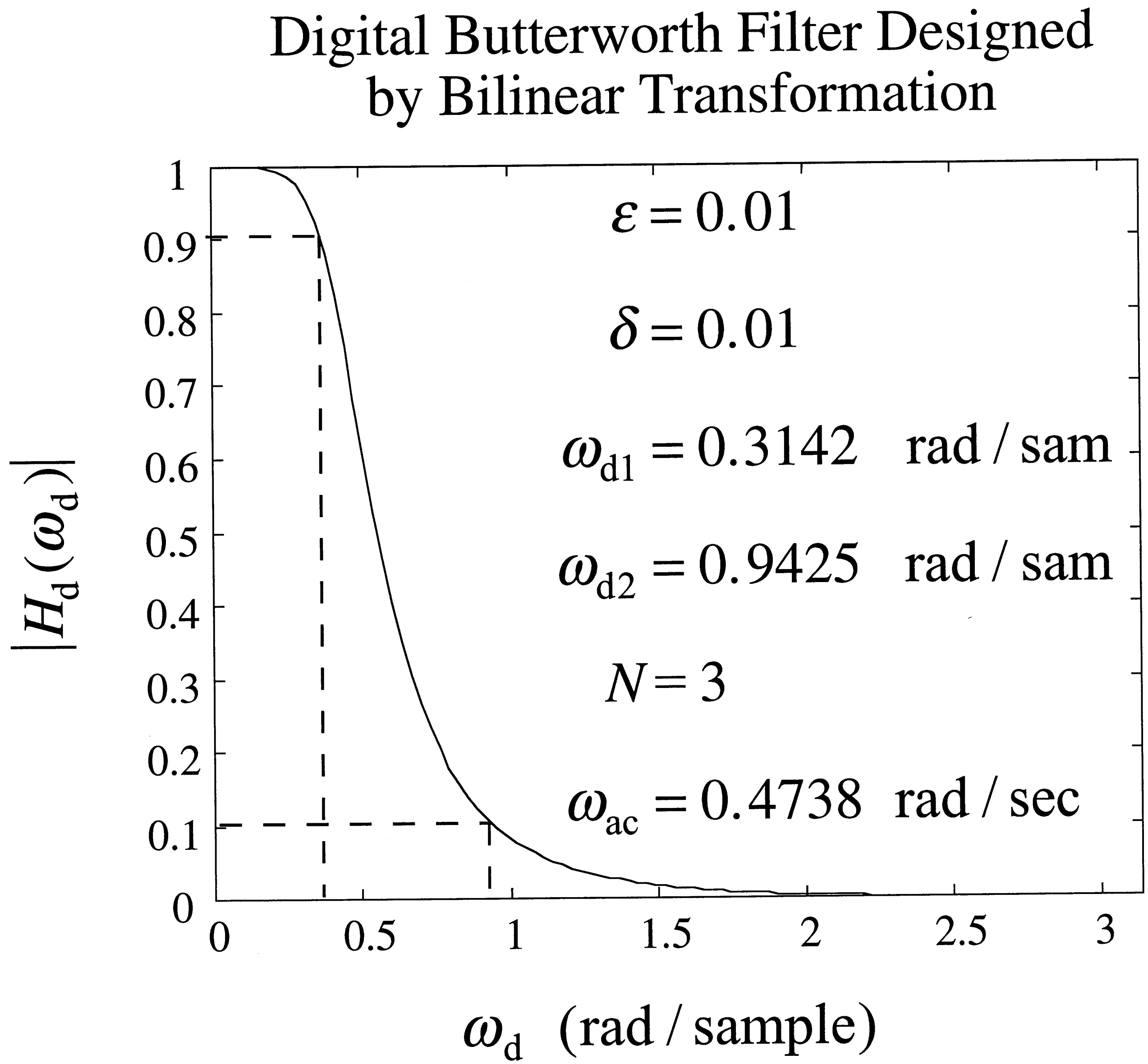
$$H_{\mathrm{d}}(z) = H_{\mathrm{a}}\left[\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right]$$

Bilinear Example No. 1 (cont.)

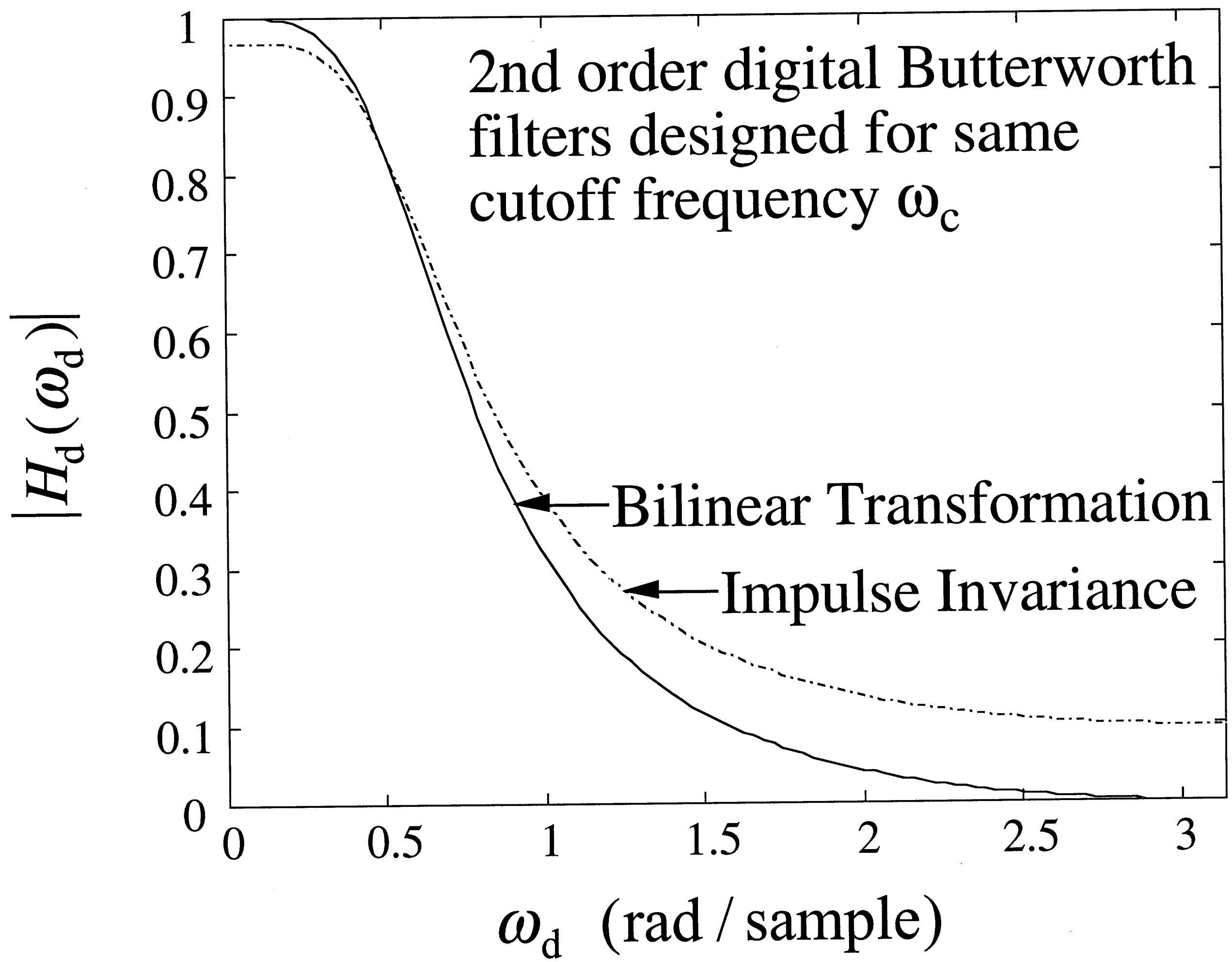
$$H_{\mathrm{d}}(z) = \frac{\omega_{\mathrm{ac}}^3\left(1+z^{-1}\right)^3}{8(1-z^{-1}-s_0)(1-z^{-1}-s_1)(1-z^{-1}-s_2)}$$

$$= \frac{0.0083 + 0.0249z^{-1} + 0.0249z^{-2} + 0.0083z^{-3}}{1.0000 - 2.0769z^{-1} + 1.5343z^{-2} - 0.3909z^{-3}}$$

Bilinear Example No. 1 (cont.)



Comparision between Bilinear Transformation and Impulse Invariance



Bilinear Example No. 2

- **Design low-pass filter** $\omega_{dc} = \pi / 5$ rad / sample
  - **cutoff frequency**
  - **transition bandwidth and ripple**

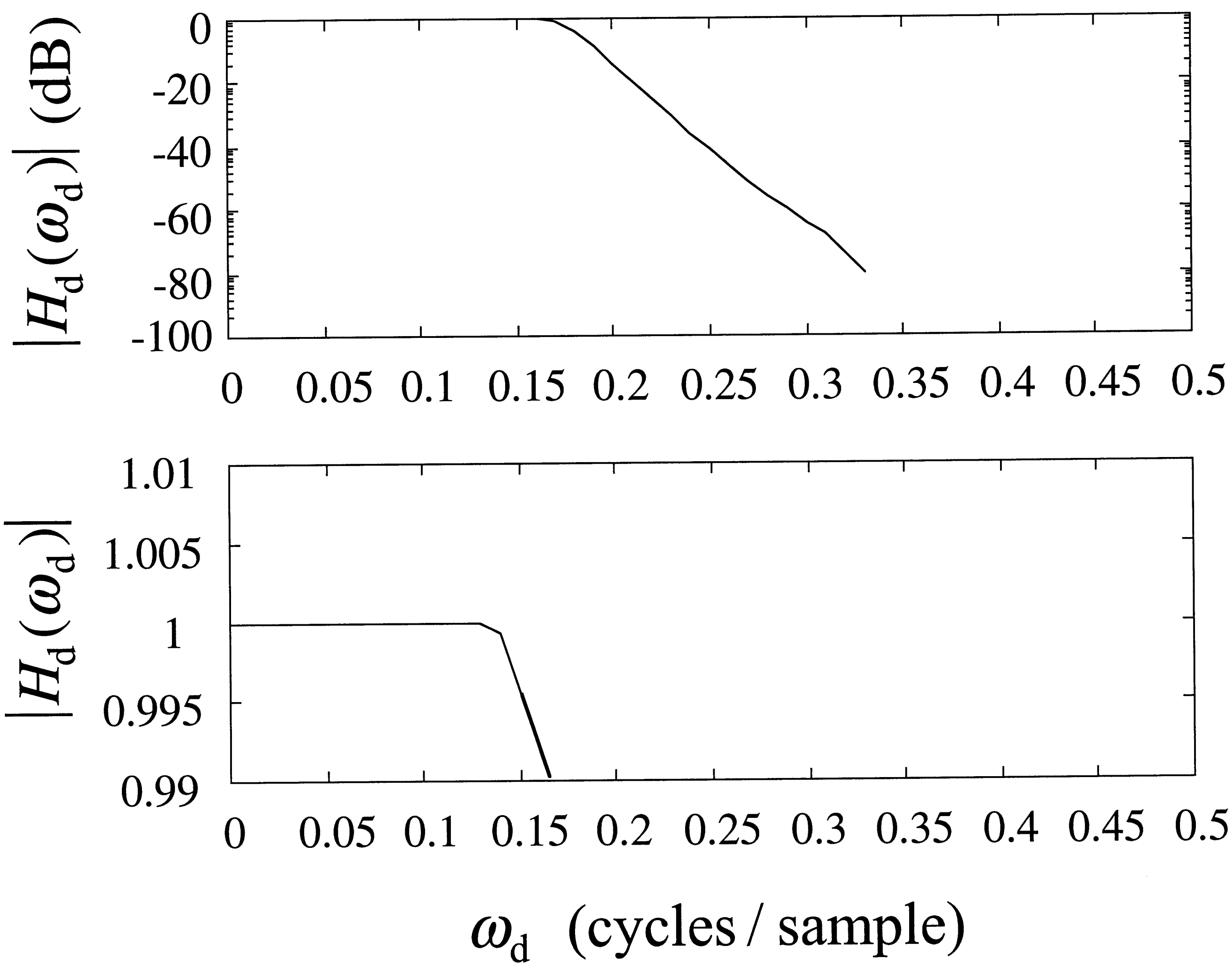
$$\begin{aligned}\Delta\omega_{d} &= 0.1\pi \text{ radians / sample} \\ &= 0.05 \text{ cycles / sample}\end{aligned}$$

$$\delta = 0.01$$

  - **use Butterworth analog prototype**
  - **set $T = 1$**

- **Result: N = 13**

Bilinear Example No. 2 (cont.)



Frequency Transformations of Lowpass IIR Filters

- **Given a prototype lowpass digital filter design with cutoff frequency θ_c , we can transform directly to:**
  - **a lowpass digital filter with a different cutoff frequency**
  - **a highpass digital filter**
  - **a bandpass digital filter**
  - **a bandstop digital filter**
- **General form of transformation**

$$H(z) = H_{\text{lp}}(Z)\Big|_{Z^{-1}=G(z^{-1})}$$

Frequency Transformations of Lowpass IIR Filters (cont.)

- **Example - lowpass to lowpass**
  - **Transformation**

$$Z^{-1} = \frac{z^{-1} - \alpha}{1 - \alpha z^{-1}}$$

  - **Associated design formula**

$$\alpha = \frac{\sin\left(\frac{\theta_c - \omega_c}{2}\right)}{\sin\left(\frac{\theta_c + \omega_c}{2}\right)}$$