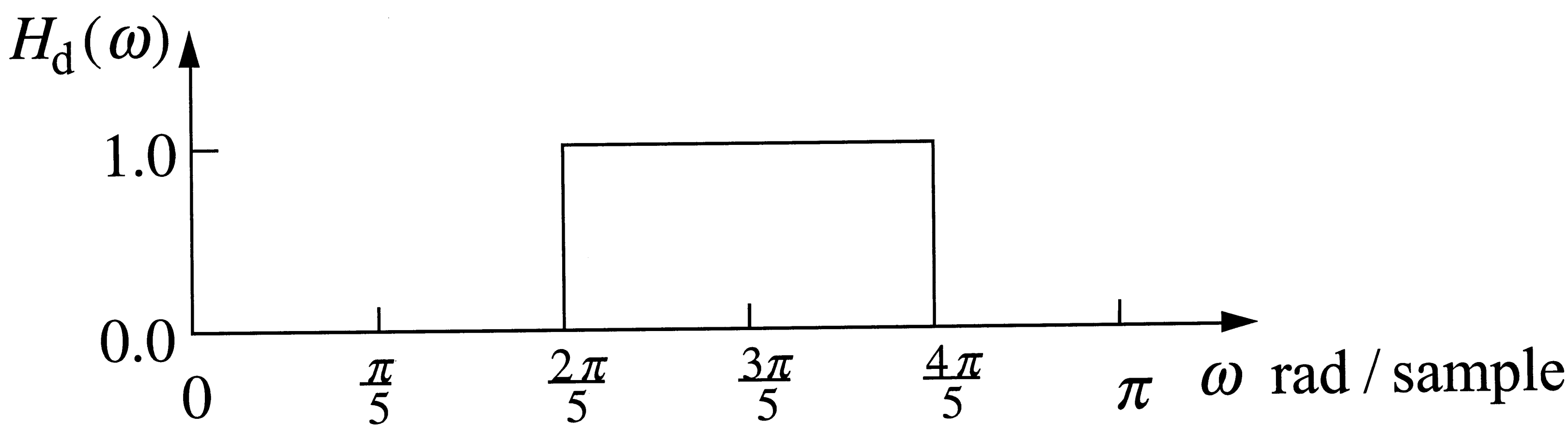
Digital Filter Design

Synopsis

- Overview of filter design problem
- Finite impulse response filter design
- Infinite impulse response filter design

Ideal Impulse Response

- **Consider same ideal frequency response as before**



- **Inverse DTFT**

$$h[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(\omega) e^{j\omega n} d\omega$$

Ideal Impulse Response (cont.)

$$h[n] = \frac{1}{2\pi}\left\{ \int_{-4\pi/5}^{-2\pi/5} e^{j\omega n} d\omega + \int_{2\pi/5}^{4\pi/5} e^{j\omega n} d\omega \right\}$$

$$h[n] = \frac{1}{j2\pi n}\left\{ \left[e^{-j2\pi n/5} - e^{-j4\pi n/5} \right] + \left[e^{j4\pi n/5} - e^{-j2\pi n/5} \right] \right\}$$

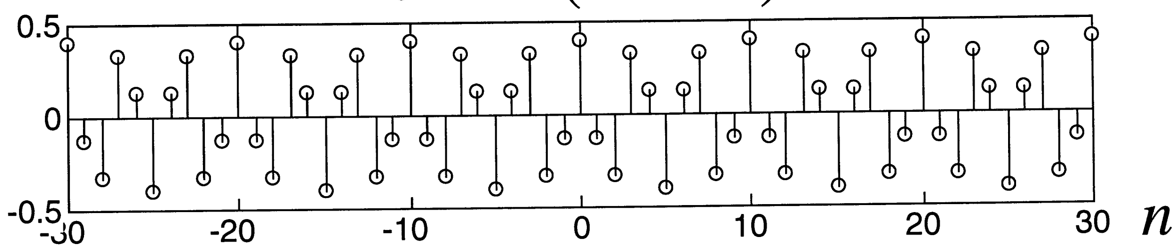
$$h[n] = \frac{1}{j2\pi n}\left\{ e^{-j3\pi n/5}\left[e^{j\pi n/5} - e^{-j\pi n/5} \right] + e^{j3\pi n/5}\left[e^{j\pi n/5} - e^{-j\pi n/5} \right] \right\}$$

$$h[n] = \left\{ e^{-j3\pi n/5}\frac{1}{5}\mathrm{sinc}\left(\frac{\mathrm{n}}{5}\right) + e^{j3\pi n/5}\frac{1}{5}\mathrm{sinc}\left(\frac{\mathrm{n}}{5}\right) \right\}$$

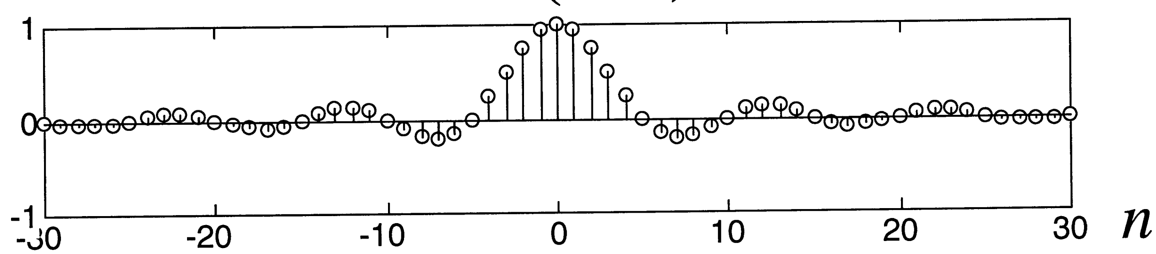
Ideal Impulse Response (cont.)

$$h[n] = 0.4\cos(3\pi n/5)\,\text{sinc}(n/5)$$

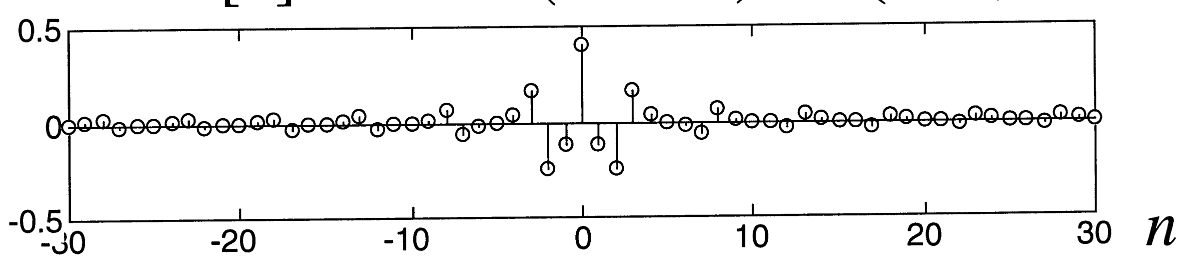
$$0.4\cos(3\pi n/5)$$



$$\text{sinc}(n/5)$$



$$h[n] = 0.4\cos(3\pi n/5)\,\text{sinc}(n/5)$$



Approximation of Desired Filter Impulse Response by Truncation

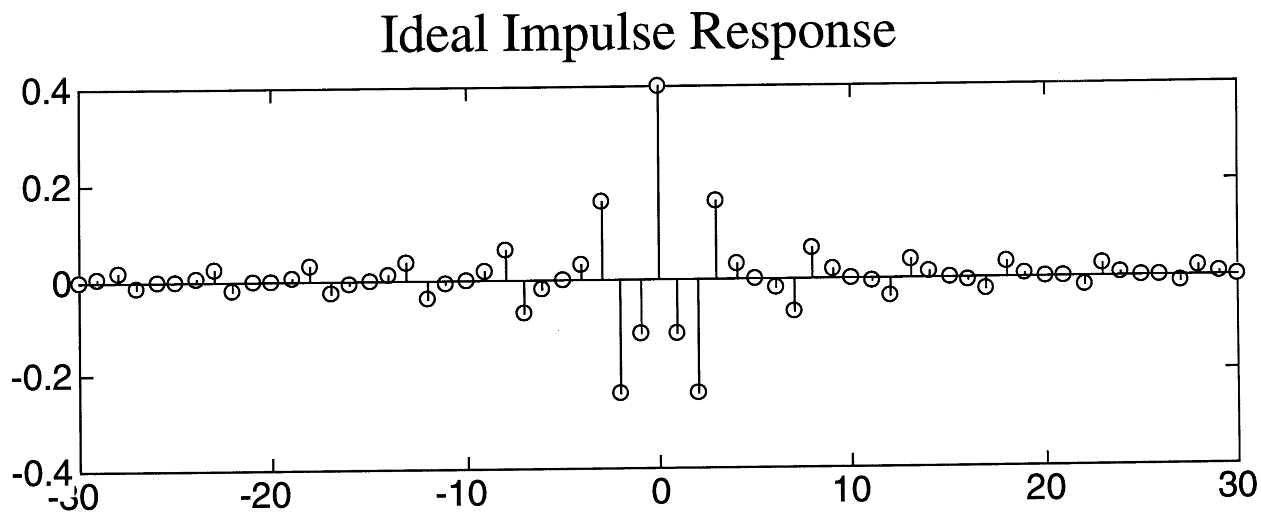
- **FIR filter equation**

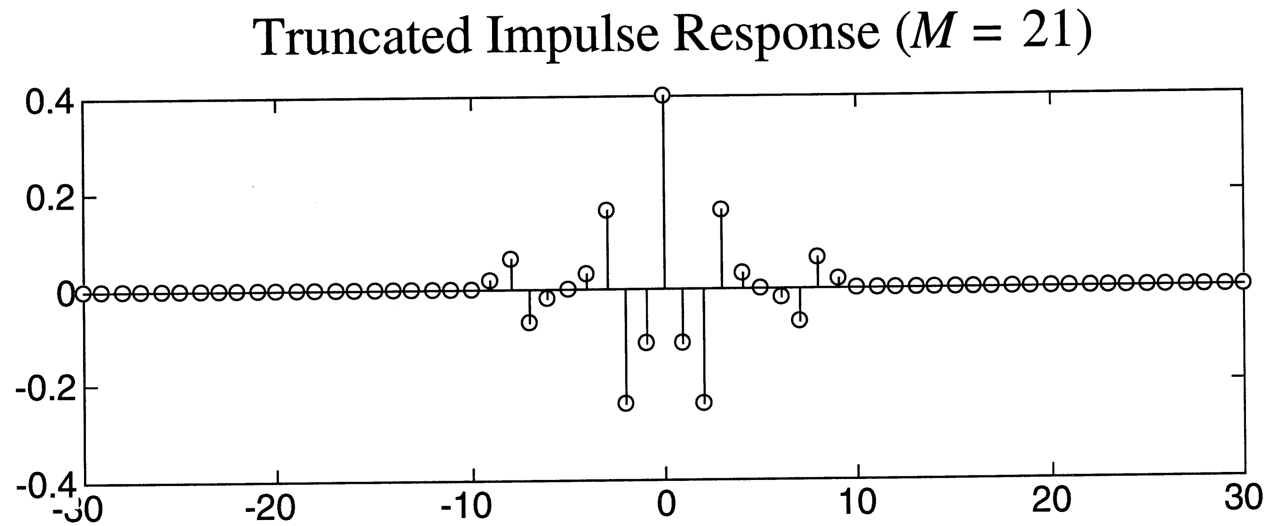
$$y[n] = \sum_{m=0}^{M-1} a_m x[n-m]$$

- **Filter coefficients (assuming *M* is odd)**

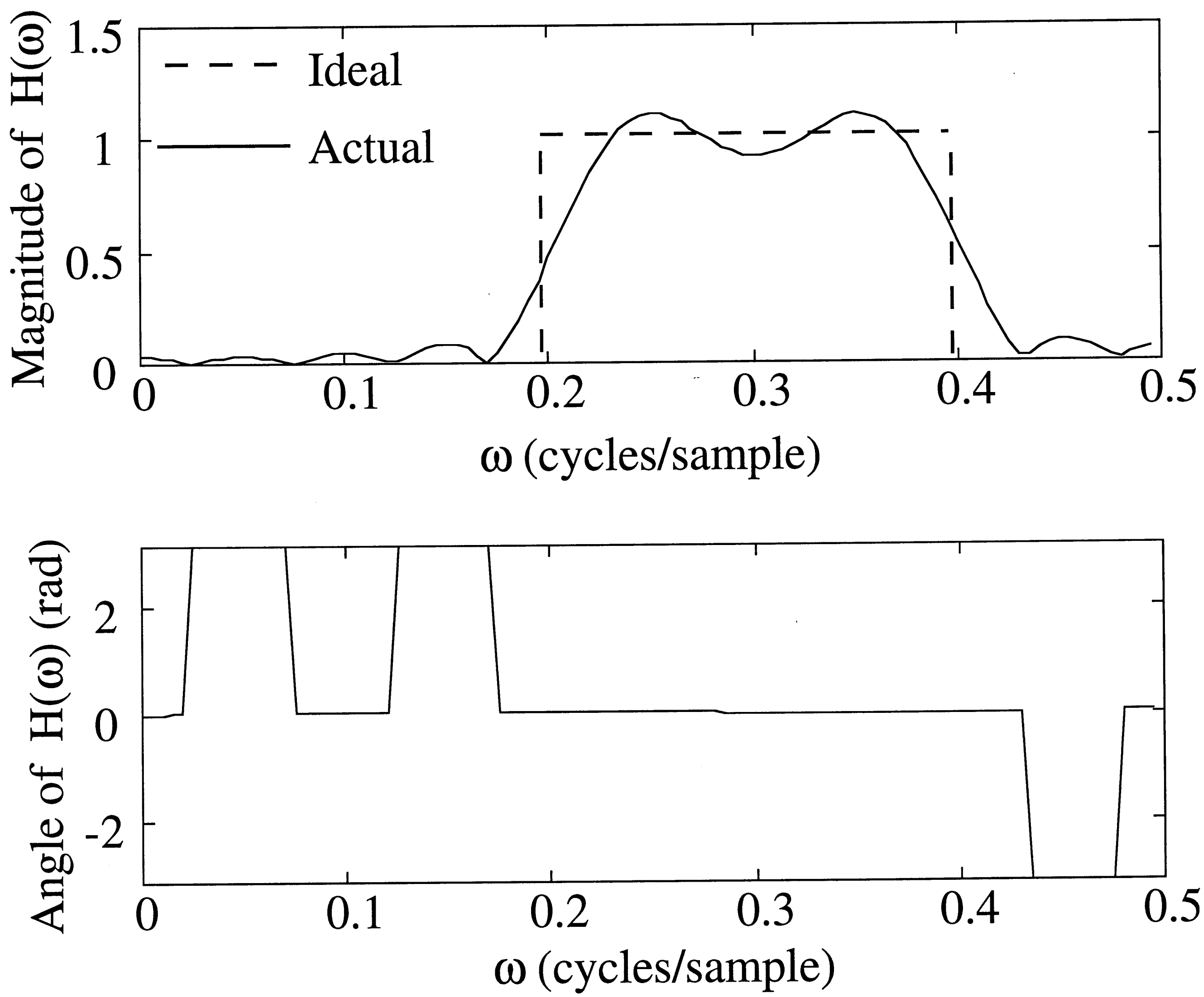
$$a_m = h_{\text{ideal}}[m-(M-1)/2], \quad m = 0,\ldots,M-1$$

Truncation of Impulse Response (cont.)

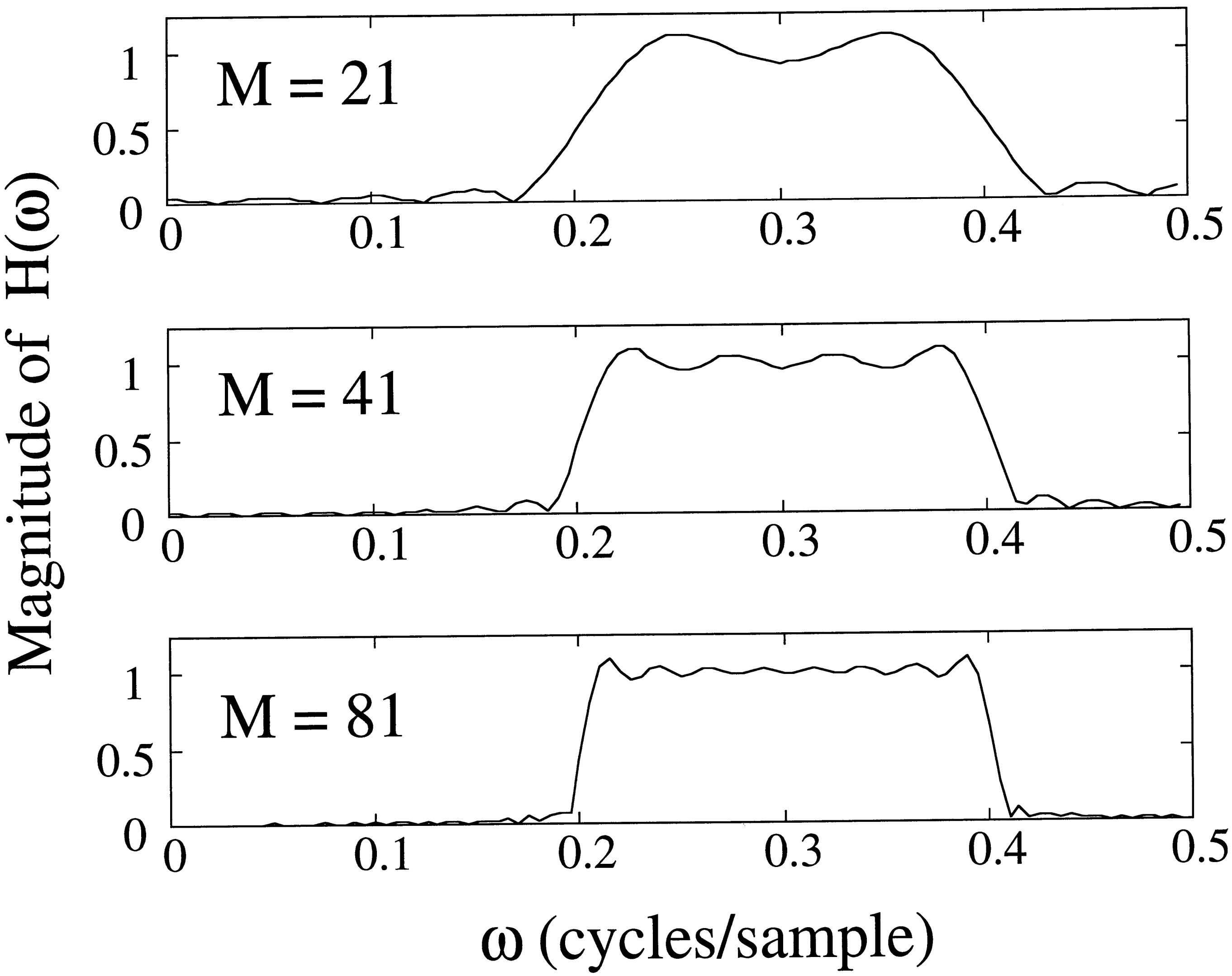




Frequency Response of Truncated Filter



Effect of Filter Length



Analysis of Truncation

- **Ideal infinite impulse response -** $h_{\text{ideal}}[n]$
- **Actual, finite impulse response -** $h_{\text{actual}}[n]$
- **Window sequence -** $w[n]$

$$w[n] = \begin{cases} 1, & |n| \leq (M-1)/2 \\ 0, & \text{else} \end{cases}$$

- **Relation between ideal and actual impulse responses**

$$h_{\text{actual}}[n] = h_{\text{ideal}}[n]\, w[n]$$

$$H_{\text{actual}}(\omega) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_{\text{ideal}}(\lambda) W(\omega - \lambda) \mathrm{d}\lambda$$

DTFT of Rectangular Window

$$W(\omega) = \sum_{n=-(M-1)/2}^{n=(M-1)/2} e^{-j\omega n}$$

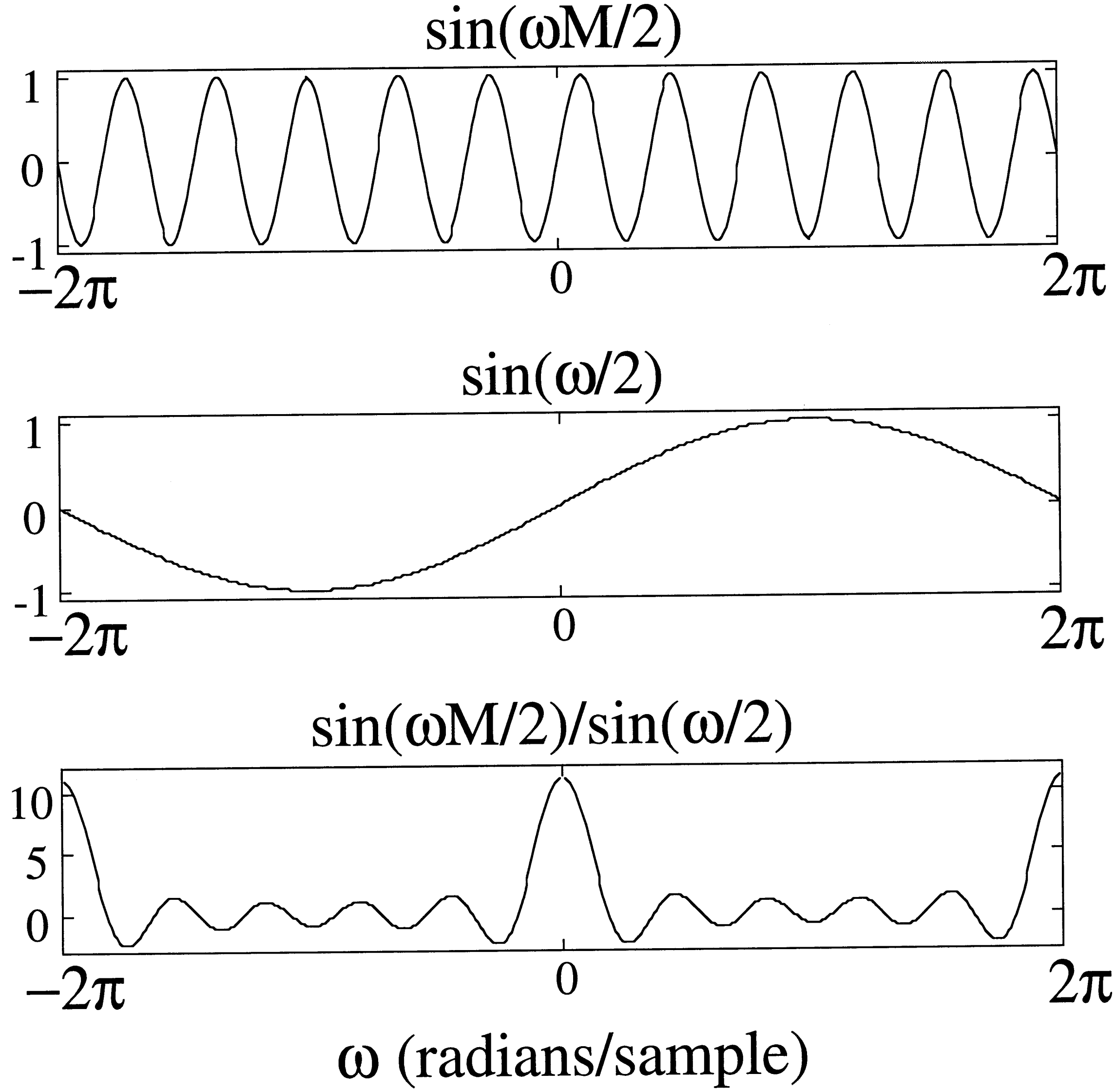
let $m = n + (M-1)/2$

$$= \sum_{m=0}^{M-1} e^{-j\omega[m-(M-1)/2]}$$

$$= e^{j\omega(M-1)/2} \frac{1-e^{j\omega M}}{1-e^{j\omega}}$$

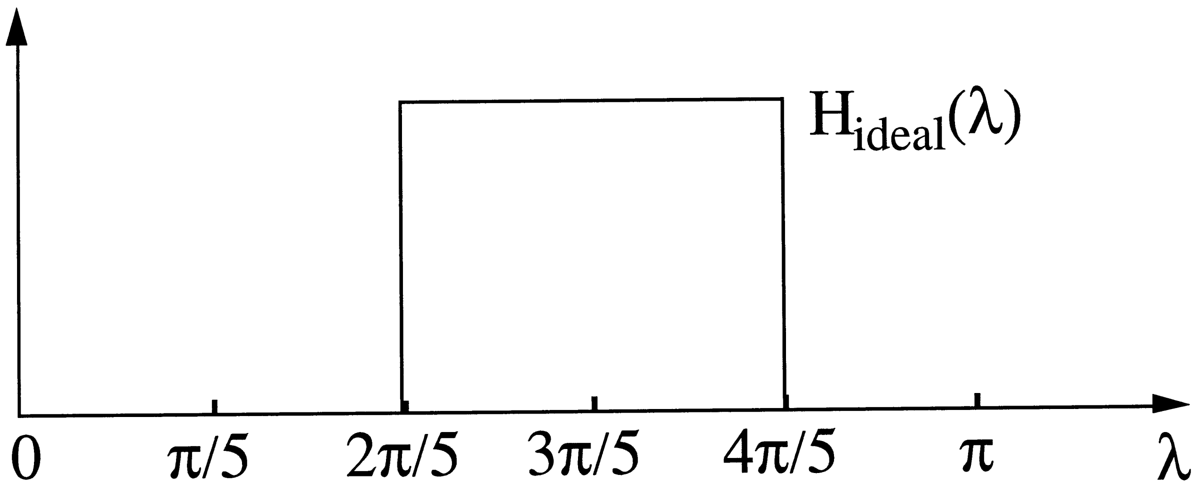
$$= \frac{\sin(\omega M/2)}{\sin(\omega/2)}$$

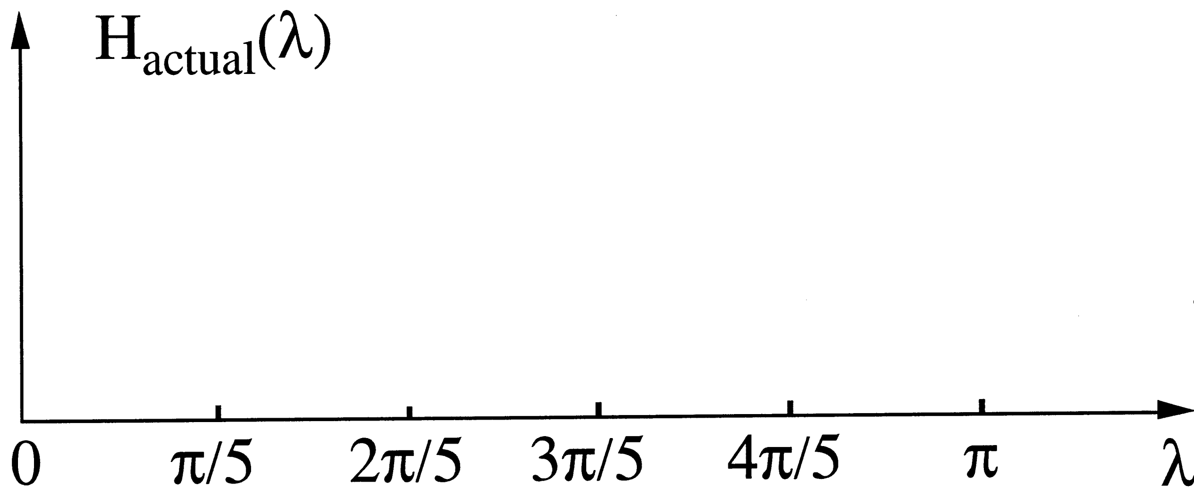
DTFT of Rectangular Window (Cont.)

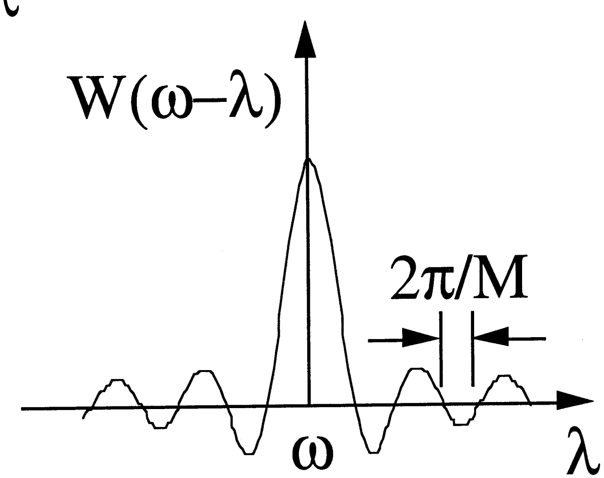


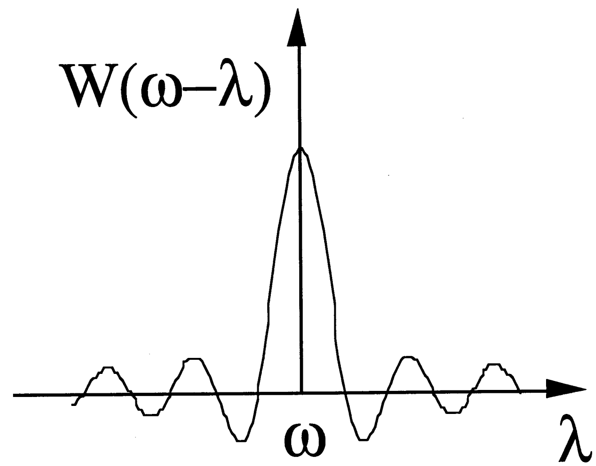
Graphical View of Convolution

$$H_{\text{actual}}(\omega) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_{\text{ideal}}(\lambda) W(\omega - \lambda) d\lambda$$

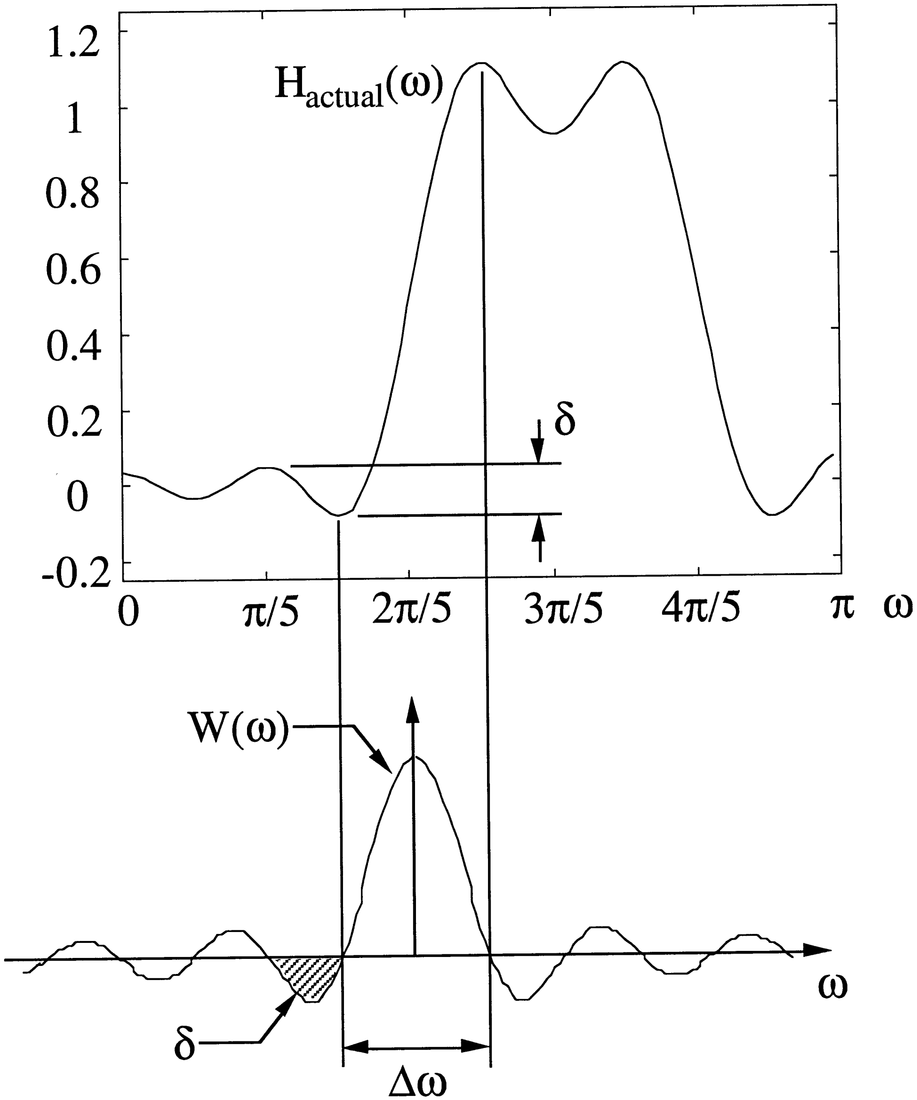







W(ω–λ)
ω
λ

Relation between Window Attributes and Filter Frequency Response



Parameter	W(ω)	$H_{actual}(\omega)$
Δω	Mainlobe width	Transition bandwidth
δ	Area of first sidelobe	Passband and stopband ripple

Design of FIR Filters by Windowing

- **Relation between ideal and actual impulse responses**

$$h_{\text{actual}}[n] = h_{\text{ideal}}[n]w[n]$$

- **Choose window sequence w[n] for which DTFT has**
  - **minimum mainlobe width**
  - **minimum sidelobe area**
- **Kaiser window is best choice**
  - **based on optimal prolate spheroidal wavefunctions**
  - **contains a parameter that permits tradeoff between mainlobe width and sidelobe area**

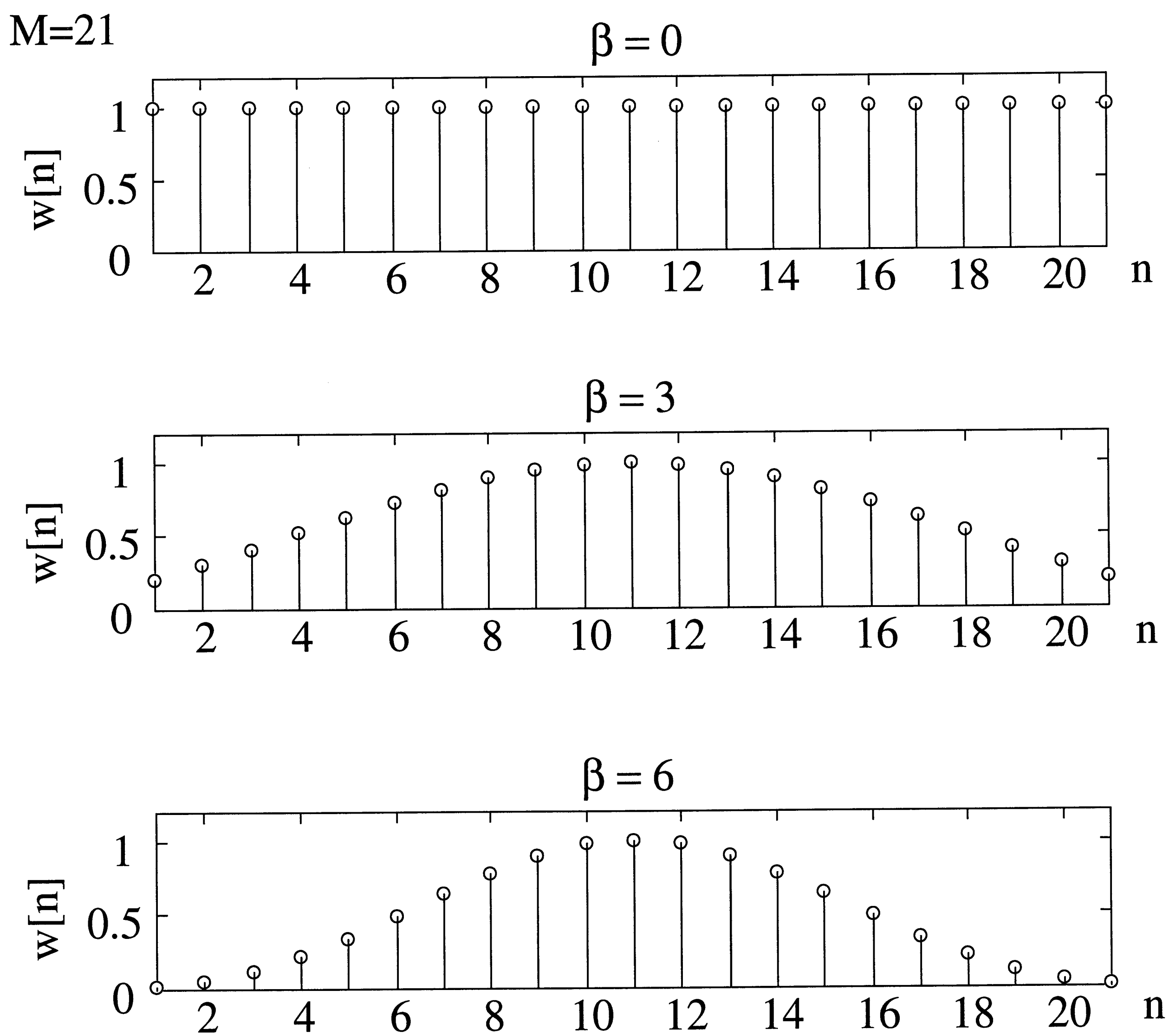
Kaiser Window

$$w[n] = \begin{cases} \dfrac{I_0[\beta(1-[(n-\alpha)/\alpha]^2)^{1/2}]}{I_0(\beta)}, & 0 \le n \le M-1 \\ 0, & \text{else} \end{cases}$$

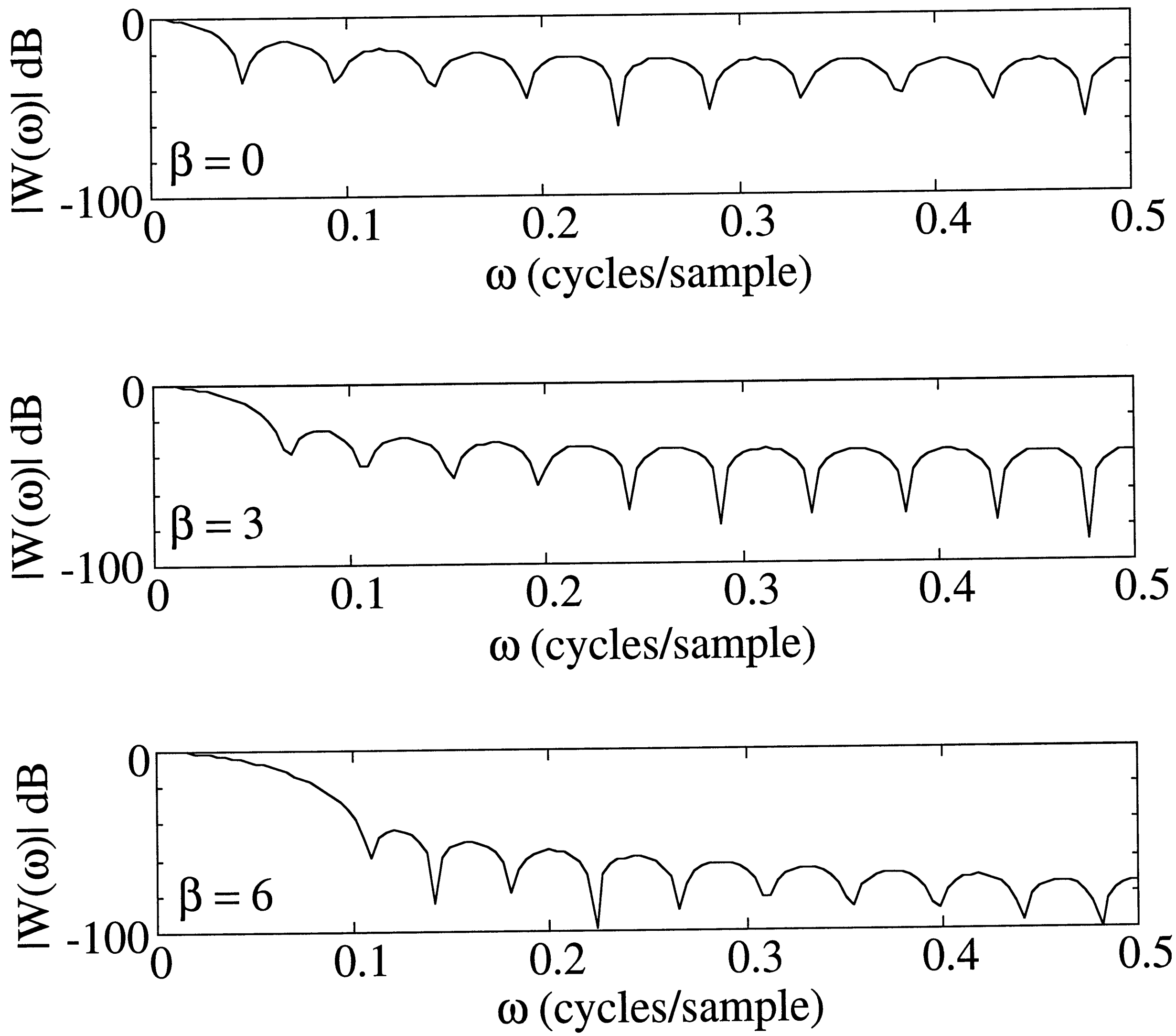
$$\alpha = (M-1)/2$$

$I_0(\cdot)$ **zeroth-order modified Bessel function of the first kind**

Effect of Parameter β

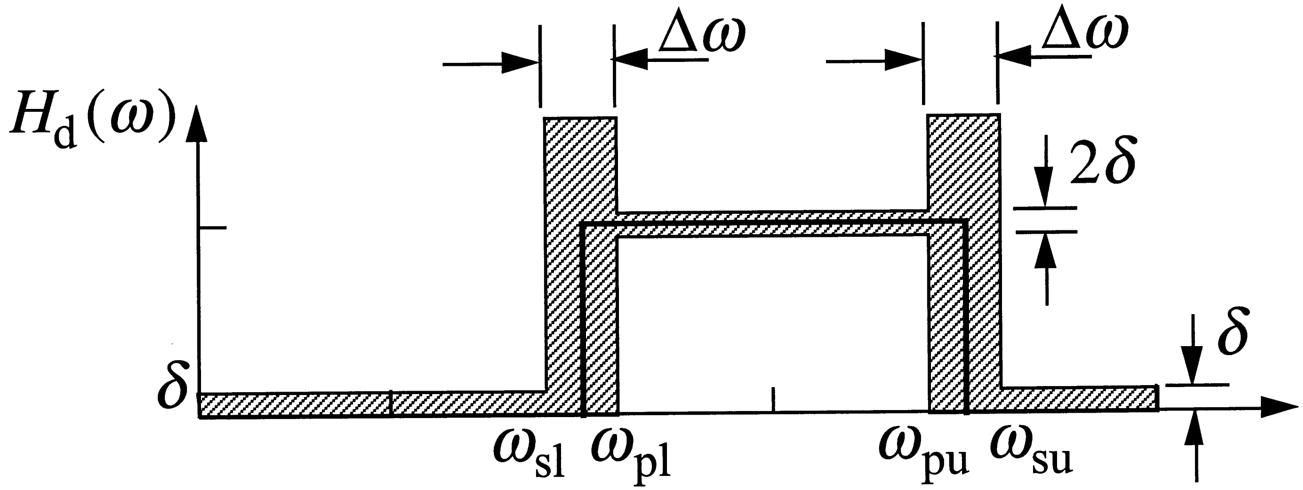


Effect of Parameter β (cont.)



Filter Design Procedure for Kaiser Window

1. Choose stopband and passband ripple and transition bandwidth



2. Determine parameter β

$$A = -20\log_{10}\delta$$

$$\beta = \begin{cases} 0.112(A-8.7), & A > 50, \\ 0.5842(A-21)^{0.4} + 0.07886(A-21), & 21 \le A \le 50, \\ 0.0, & A < 21. \end{cases}$$

Filter Design Procedure for Kaiser Window (cont.)

3. Determine filter length *M*

$$M = \frac{A-8}{2.285\Delta\omega} + 1$$

- **Example**

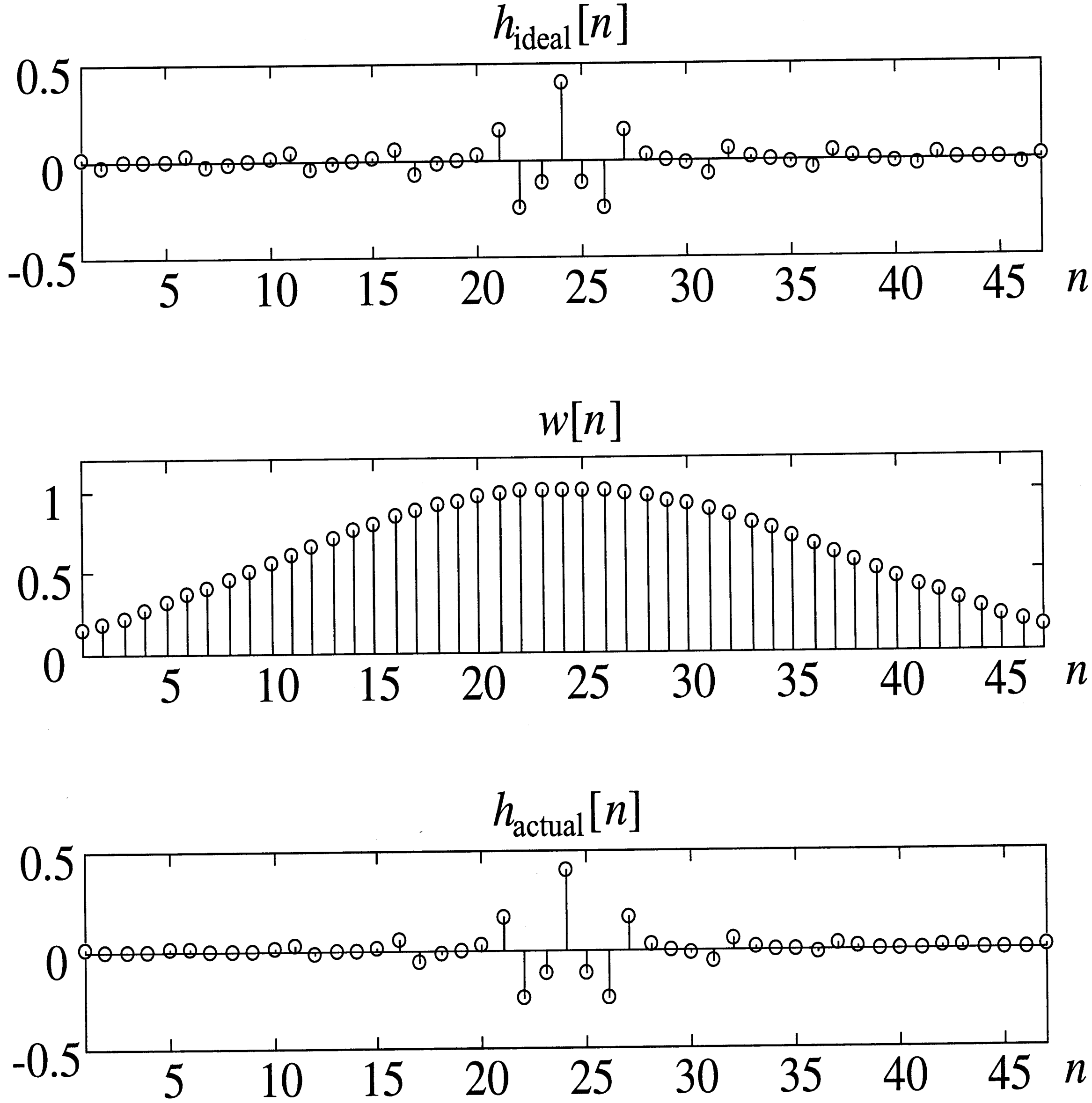
$$\begin{aligned}\Delta\omega &= 0.1\pi \ \text{ radians / sample} \\ &= 0.05 \ \text{ cycles / sample}\end{aligned}$$

$$\delta = 0.01$$

$$\beta = 3.4$$

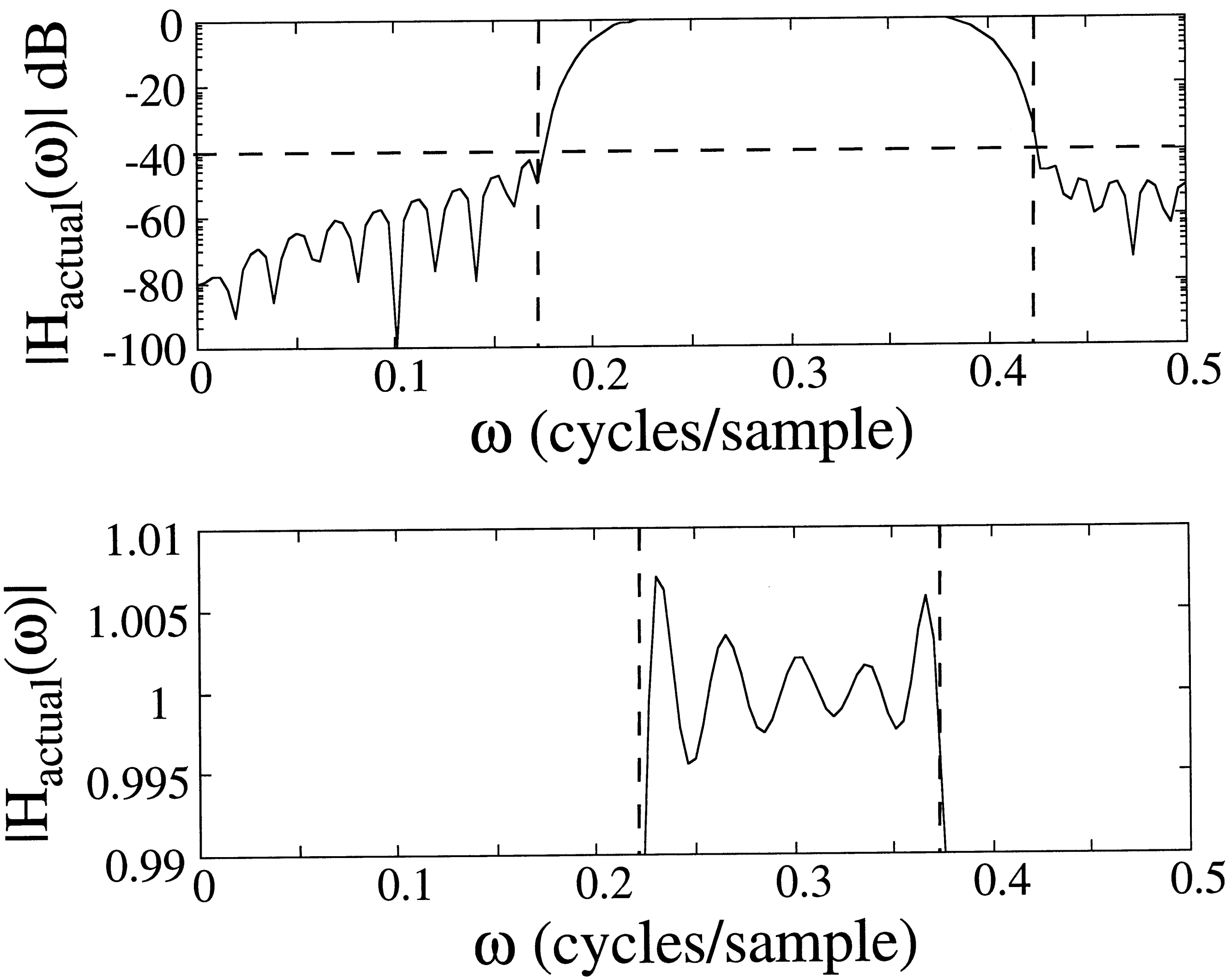
$$M = 45.6$$

Filter Designed with Kaiser Window



$M = 47$
$\beta = 3.4$

Kaiser Filter Frequency Response



$M = 47$

$\beta = 3.4$

$\delta = 0.01$

$= -40 \text{ dB}$

$\Delta\omega = 0.05$

$\omega_{sl} = 0.175$

$\omega_{pl} = 0.225$

$\omega_{pu} = 0.375$

$\omega_{su} = 0.425$

Optimal FIR Filter Design

- **Although the Kaiser window has certain optimal properties, filters designed using it are not optimal in any sense.**
- **Consider the design of filters to minimize integral mean-squared frequency error**
- **Problem:**

Choose $h_{\text{actual}}[n], n = -(M-1)/2, \ldots, (M-1)/2$

to minimize

$$E = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| H_{\text{actual}}(\omega) - H_{\text{ideal}}(\omega) \right|^2 \mathrm{d}\omega$$

Minimum Mean-Squared Error Design

- **Using Parseval's relation, we obtain a direct solution to this problem**

$$E = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|H_{\text{actual}}(\omega) - H_{\text{ideal}}(\omega)\right|^2 \mathrm{d}\omega$$

$$= \sum_{n=-\infty}^{\infty} \left|h_{\text{actual}}[n] - h_{\text{ideal}}[n]\right|^2$$

$$= \sum_{n=-(M-1)/2}^{(M-1)/2} \left|h_{\text{actual}}[n] - h_{\text{ideal}}[n]\right|^2 +$$

$$\sum_{n=-\infty}^{-(M+1)/2} \left|h_{\text{ideal}}[n]\right|^2 + \sum_{n=(M+1)/2}^{\infty} \left|h_{\text{ideal}}[n]\right|^2$$

Minimum Mean-Squared Error Design (cont.)

- **Solution**

  set $h_{\text{actual}}[n] = h_{\text{ideal}}[n],\ n = -(M-1)/2, \ldots, (M-1)/2$

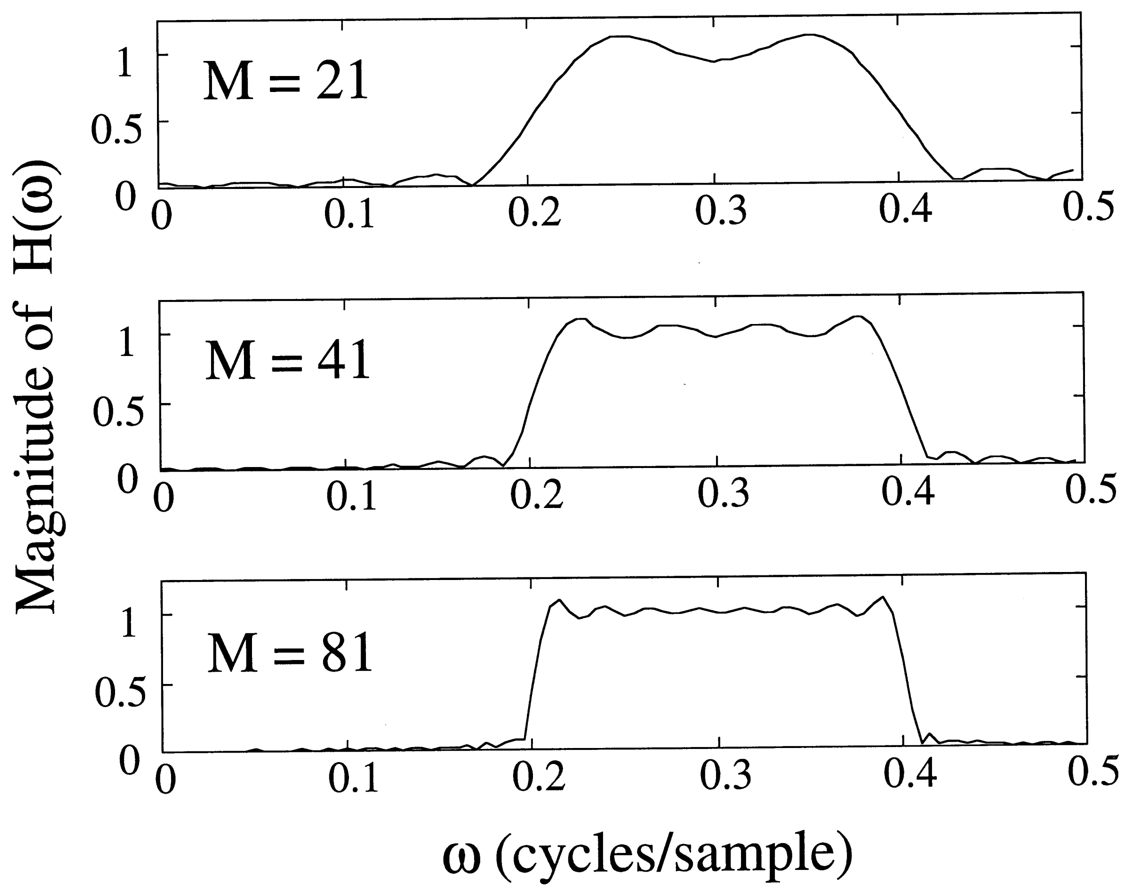
  minimum error is given by

$$E = \sum_{n=-\infty}^{-(M+1)/2} \left| h_{\text{ideal}}[n] \right|^2 + \sum_{n=(M+1)/2}^{\infty} \left| h_{\text{ideal}}[n] \right|^2$$

- **But this is nothing more than truncation of the ideal impulse response**
- **Conclude that original criteria of minimizing mean-squared error was not a good choice**

Minimum Mean-Squared Error Design (cont.)

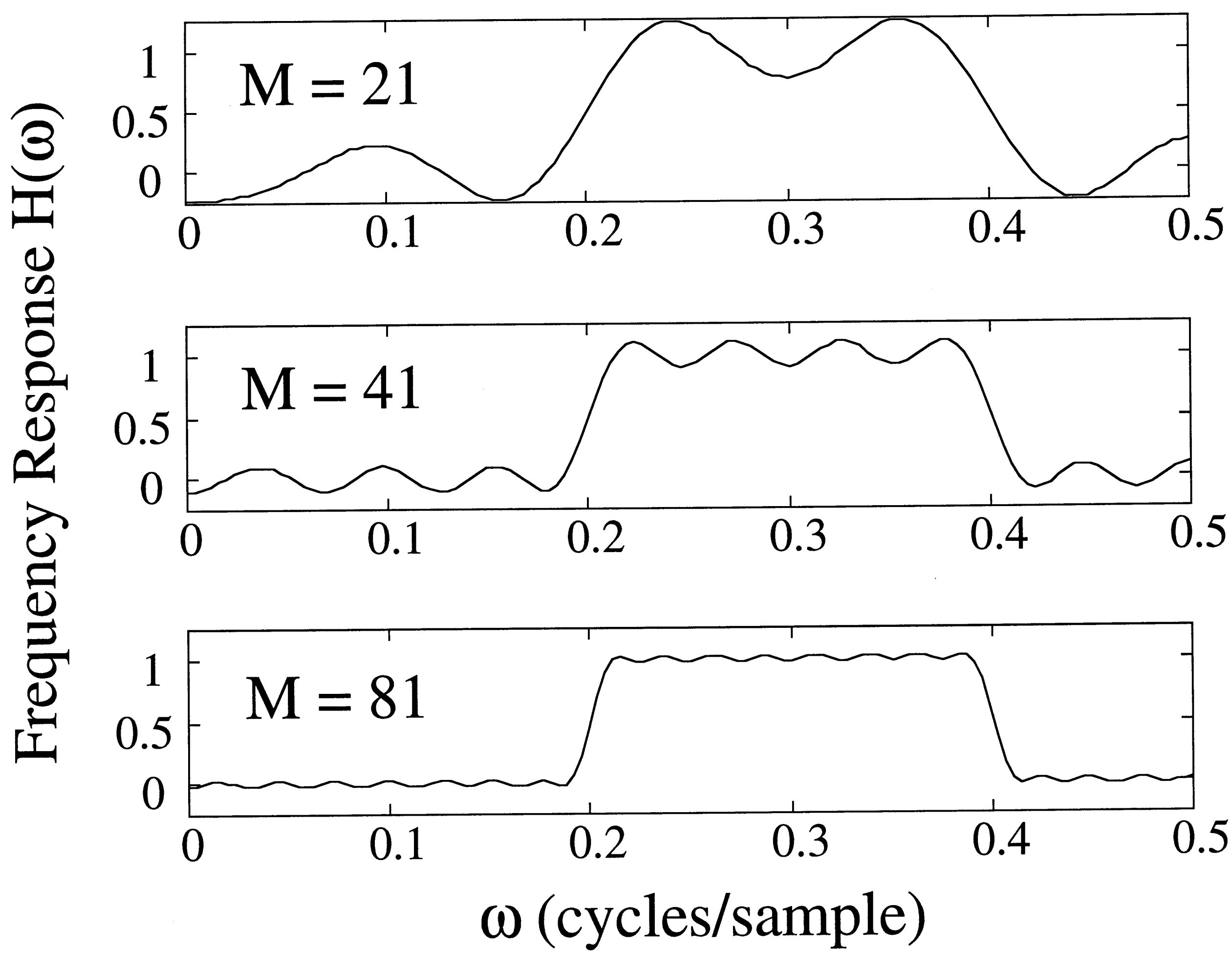
- **What was the problem with truncating the ideal impulse response?**



Minimax (Equi-ripple) Filter Design

- No matter how large we pick M for the truncated impulse response, we cannot reduce the peak ripple
- Also, the ripple is concentrated near the band edges
- Rather than focusing on the integral of the error, let's consider the maximum error
- Parks and McClellan developed a method for design of FIR filters based on minimizing the maximum of the weighted frequency domain error

Effect of Filter Length: Parks-McClellan Equiripple Filters



Comparison: Kaiser Window-Based and Parks-McClellan Filters