## 1.6.5 PERIODIC CONVOLUTION

We developed the DFT as a computable spectral representation for DT signals.

However, the existence of a very efficient algorithm for calculating it suggests another application.

Consider the filtering of a length N signal x(n) with an FIR filter containing M coefficients, where $M \ll N$.

$$y(n) = \sum_{m=0}^{M-1} h(m)\, x(n-m)$$

Computation of each output point requires M multiplications and M − 1 additions.

Based on the notion that convolution in the time domain corresponds to multiplication in the frequency domain, we perform the following computations:

1. Compute $X^{(n)}(k)$ $\quad N \log_2 N$ CO's

2. Extend h(n) with zeros to length N and compute $H^{(N)}(k)$ $\quad N \log_2 N$ CO's

3. Multiply the DFT's

$$Y^{(N)}(k) = H^{(N)}(k)\, X^{(N)}(k)$$ N complex multiplications

4. Compute inverse DFT

| | |
|---|---|
| $\tilde{y}(n) = DFT^{-1}\{Y^{(N)}(k)\}$ | $N \log_2 N$ CO's |
| Total | $3 N \log_2 N$ CO's + N complex multiplications |

The computation/output point is only $3 \log_2 N$ CO's + 1 complex multiplication.

# How Periodic Convolution Arises

- **Consider inverse DFT of product** $Y[k] = H[k]\, X[k]$

$$y[n] = \frac{1}{N} \sum_{k=0}^{N-1} H[k]\, X[k]\, e^{j 2\pi k n / N}$$

- **Substitute for** $X[k]$ **in terms of** $x[n]$

$$y[n] = \frac{1}{N} \sum_{k=0}^{N-1} H[k] \left\{ \sum_{m=0}^{N-1} x[m]\, e^{-j 2\pi k m / N} \right\} e^{j 2\pi k n / N}$$

# How Periodic Convolution Arises (Cont.)

- **Interchange order of summations**

$$y[n] = \sum_{m=0}^{N-1} x[m] \left\{ \frac{1}{N} \sum_{k=0}^{N-1} H[k]\, e^{-j2\pi k(n-m)/N} \right\}$$

$$= \sum_{m=0}^{N-1} x[m]\, h[n-m]$$

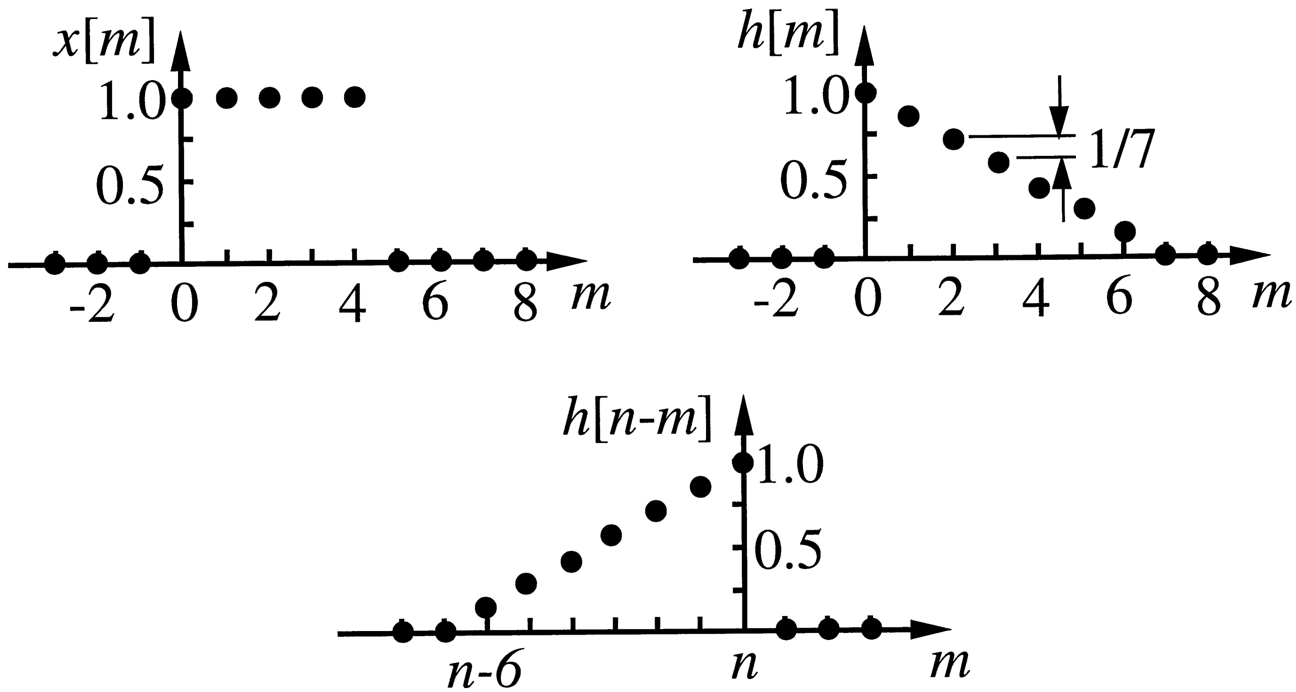
- **Note periodicity of sequence** $h[n]$

$$h[n-m] = h[(n-m) \bmod \mathrm{N}]$$

# Periodic vs. Aperiodic Convolution

- **Aperiodic Convolution**

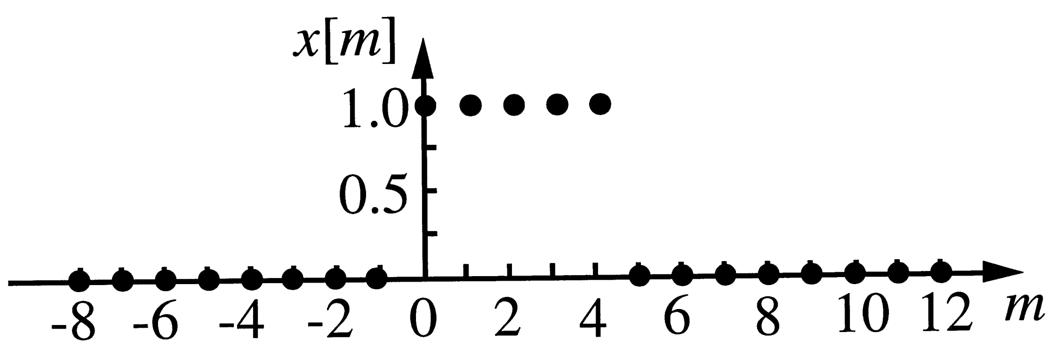
$$y[n] = h[n] * x[n] = \sum_{m=-\infty}^{\infty} h[n-m]\, x[m]$$

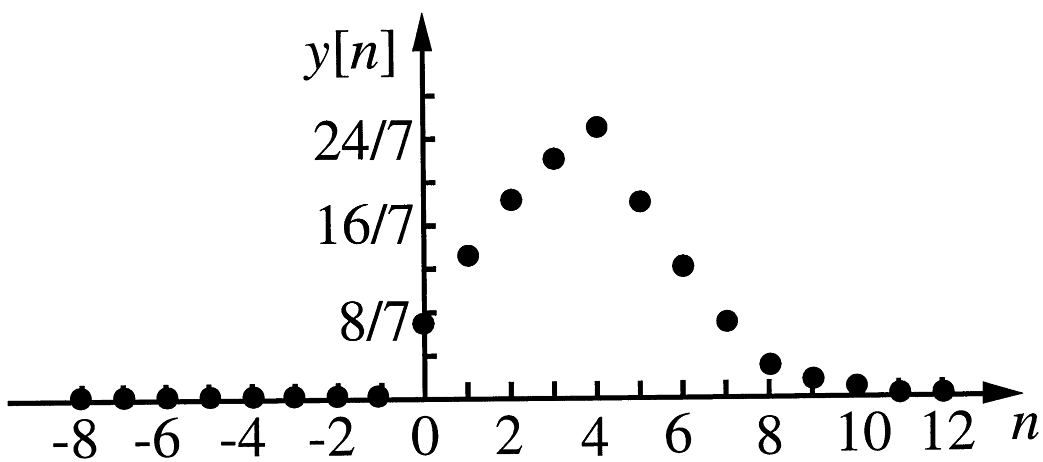


# Periodic vs. Aperiodic Convolution

- **Aperiodic Convolution (cont.)**

$$y[n] = h[n] * x[n] = \sum_{m=-\infty}^{\infty} h[n-m]\, x[m]$$

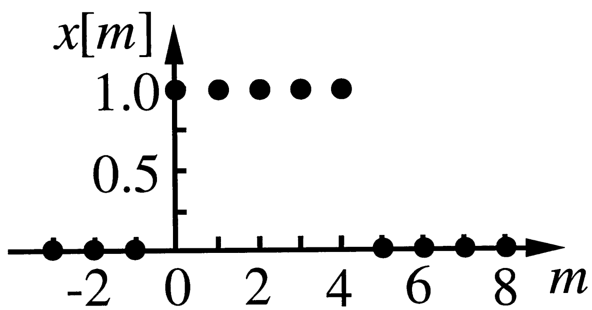


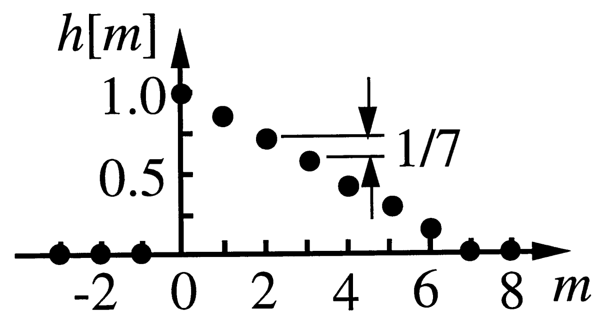



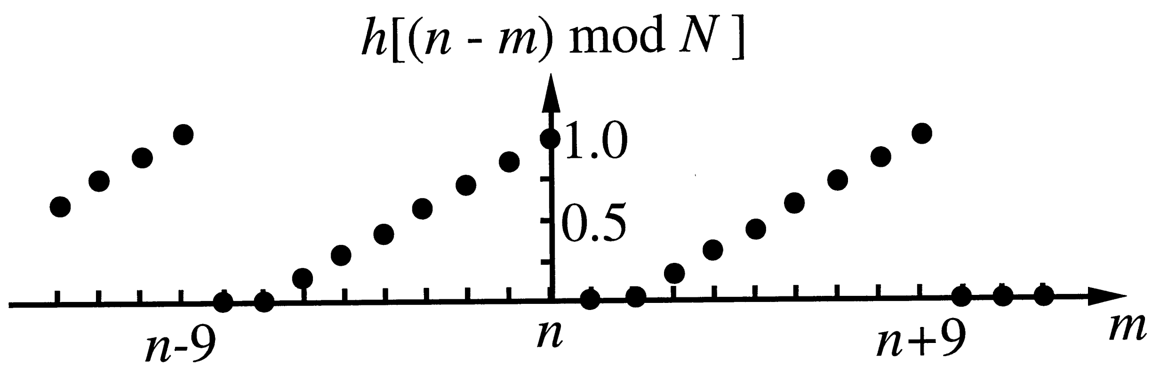
# Periodic vs. Aperiodic Convolution

- **Periodic Convolution** $(N = 9)$

$$y[n] = h[n] \circledast x[n] = \sum_{m=0}^{N-1} h[(n-m) \bmod \mathrm{N}]\, x[m]$$



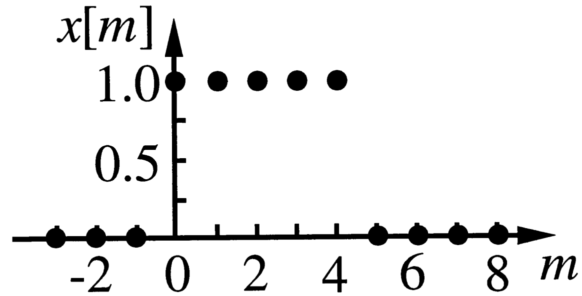



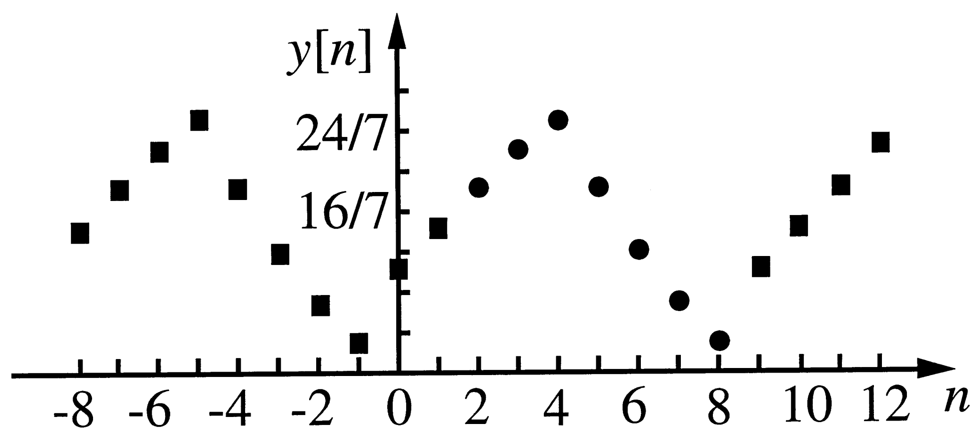



# Periodic vs. Aperiodic Convolution

- **Periodic Convolution (cont.)** $(N = 9)$

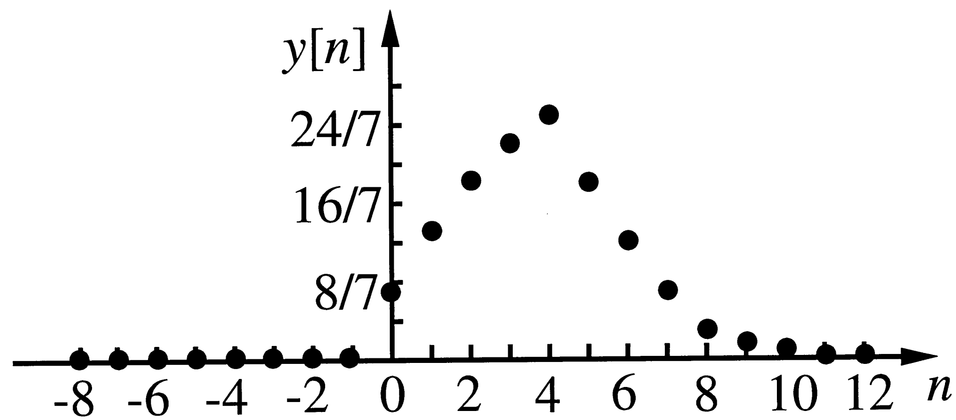
$$y[n] = h[n] \circledast x[n] = \sum_{m=0}^{N-1} h[(n-m) \bmod \mathrm{N}]\, x[m]$$



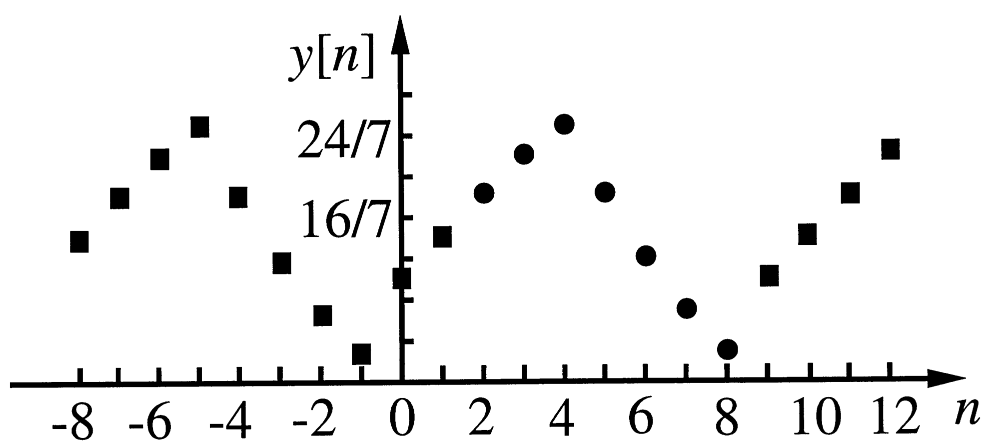



# Comparison

- **Aperiodic Convolution**

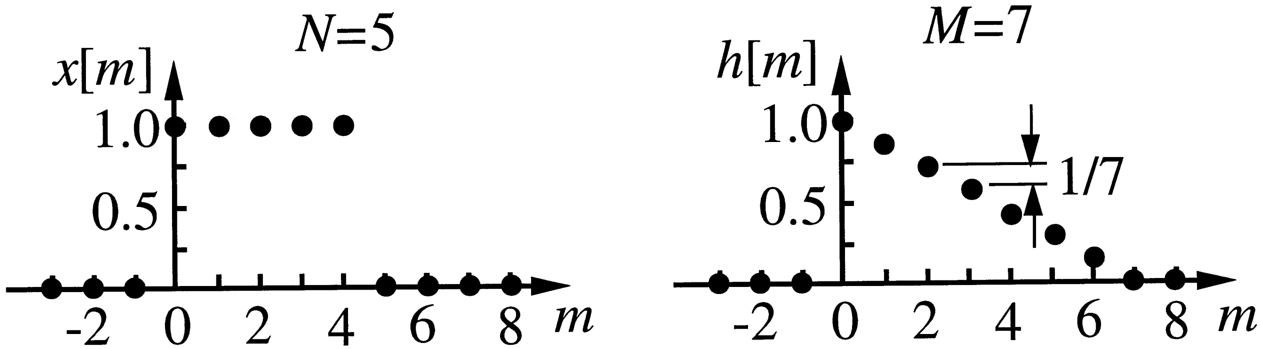


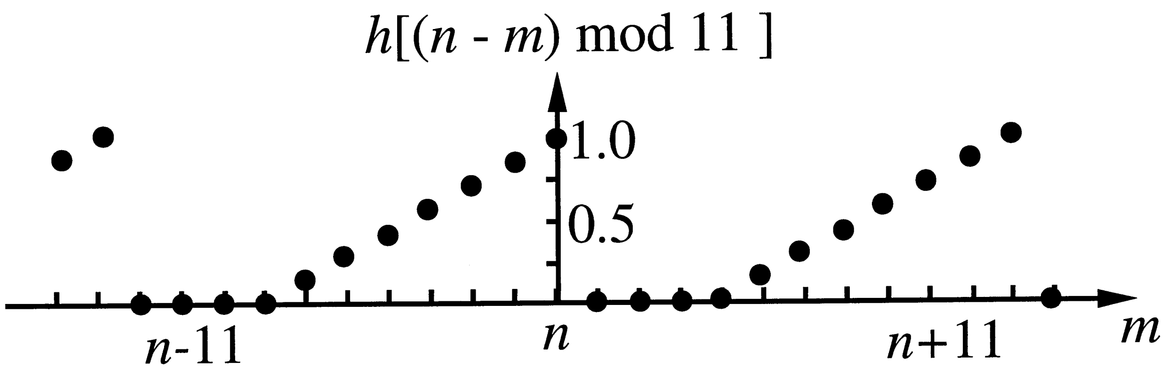

- **Periodic Convolution** $(N = 9)$



# Zero Padding to Match Aperiodic Convolution

- **Suppose $x[n]$ has length $N$ and $h[n]$ has length $M$. Periodic convolution with length $M+N-1$ will match aperiodic result.**



# Zero Padding to Match Aperiodic Convolution (cont.)