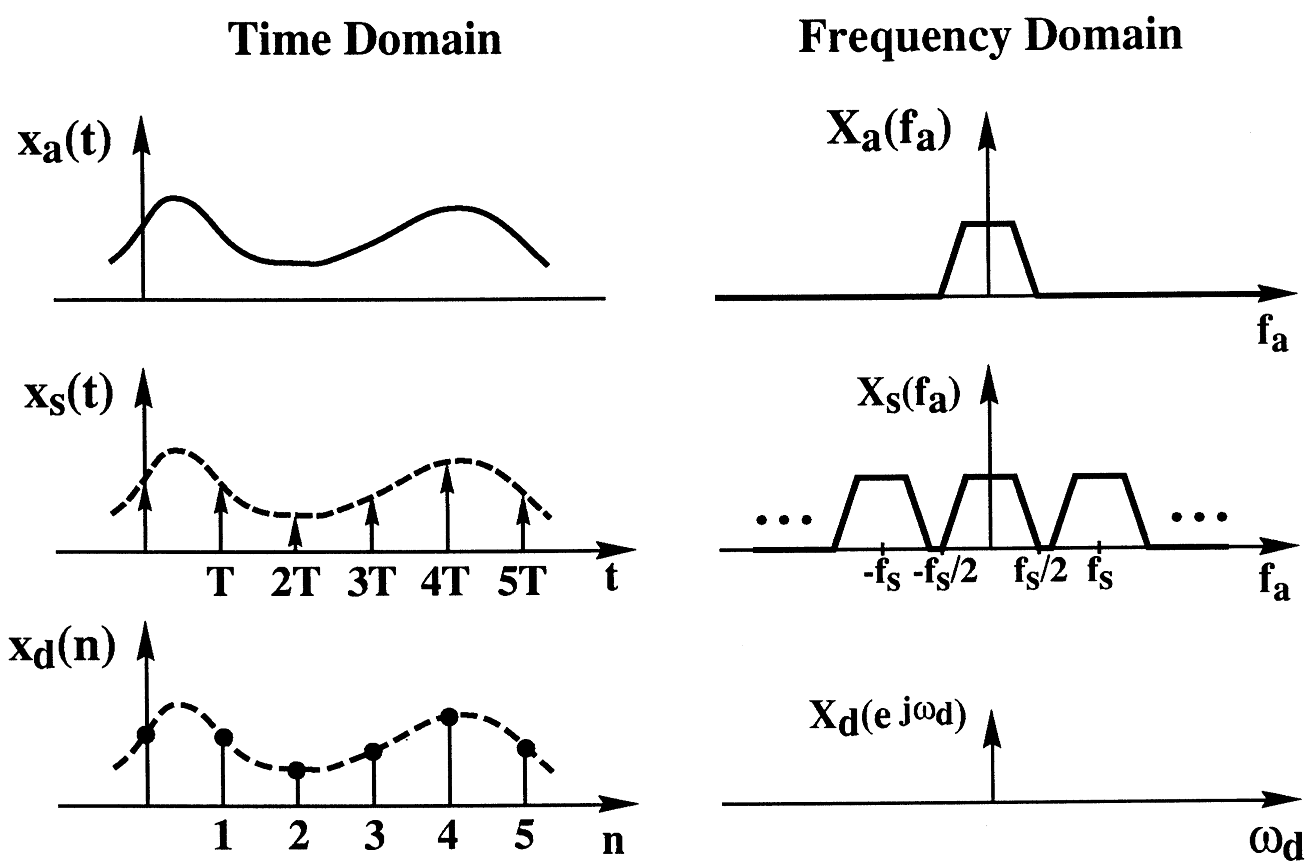
1.4.2 RELATION BETWEEN CTFT AND DTFT



We have already shown that

$$X_s(f_a) = \frac{1}{T} \text{ rep } \frac{1}{T} [X(f_a)]$$

Suppose we evaluate CTFT of $x_s(t)$ directly

$$X_s(f_a) = \mathscr{F}\{\sum_n x_a(nT)\, \delta(t - nT)\}$$

$$= \sum_n x_a(nT)\, \mathscr{F}\{\delta(t - nT)\}$$

$$X_s(f_a) = \sum_n x_a(nT)\, e^{-j2\pi f_a nT}$$

Recall

$$X_d(e^{j\omega_d}) \triangleq \sum_n x_d(n)\, e^{-j\omega_d n}$$

But $x_d(n) = x_a(nT)$

Let $\omega_d = 2\pi\, f_a T = 2\pi\, (f_a / f_s)$

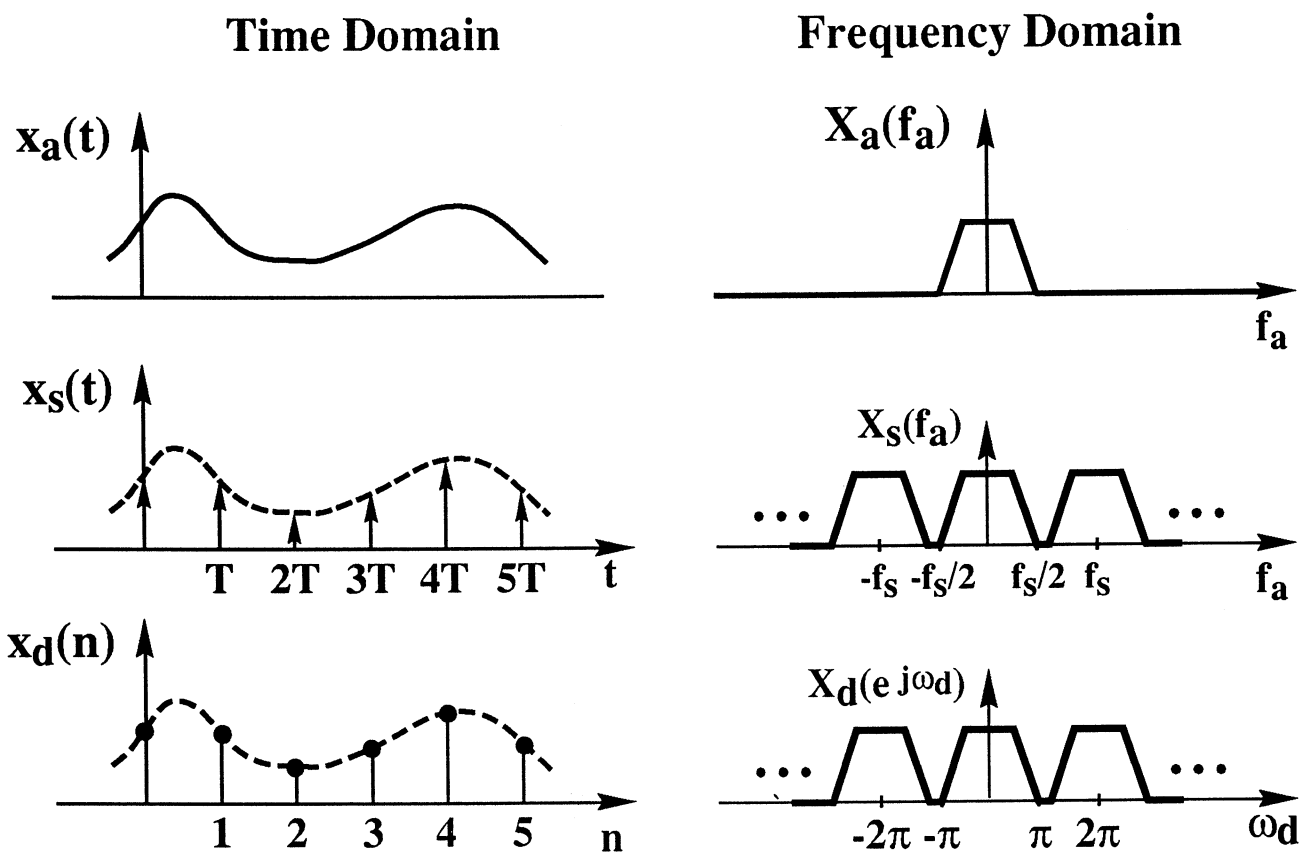
f_a	0	$f_s/2$	f_s	Hz
ω_d	0	π	2π	rad./sample

Check units

$$\omega_d \left(\frac{\text{rad.}}{\text{sample}} \right) = 2\pi \left(\frac{\text{rad.}}{\text{cycle}} \right) \frac{f_a \left(\frac{\text{cycles}}{\text{sec.}} \right)}{f_s \left(\frac{\text{samples}}{\text{sec.}} \right)}$$

Rearranging as $f_a = \left(\frac{\omega_d}{2\pi}\right) f_s$ we obtain

$$X_d(e^{j\omega_d}) = X_s\left[\left(\frac{\omega_d}{2\pi}\right) f_s\right]$$


Time Domain
Frequency Domain
$x_a(t)$
$X_a(f_a)$
f_a
$x_s(t)$
T 2T 3T 4T 5T t
$X_s(f_a)$
$-f_s$ $-f_s/2$ $f_s/2$ f_s
f_a
$x_d(n)$
1 2 3 4 5 n
$X_d(e^{j\omega_d})$
-2π $-\pi$ π 2π
ω_d

Example

1. CT Analysis

$$x_a(t) = \cos(2\pi f_{a0} t)$$

$$X_a(f_a) = \frac{1}{2}[\delta(f_a - f_{a0}) + \delta(f_a + f_{a0})]$$

$$X_s(f_a) = f_s \; \text{rep}_{f_s}[X_a(f_a)]$$

$$= \frac{f_s}{2} \sum_k [\delta(f_a - f_{a0} - kf_s) + \delta(f_a + f_{a0} - kf_s)]$$

2. DT Analysis

$$x_d(n) = x_a(nT)$$

$$= \cos(2\pi f_{a0} nT)$$

$$= \cos(\omega_{d0} n)$$

$$\omega_{d0} = 2\pi f_{a0} T = 2\pi\ (f_{a0}/f_s)$$

$$X(e^{j\omega_d}) = \pi \sum_k [\delta(\omega_d - \omega_{d0} - 2\pi k) + \delta(\omega_d + \omega_{d0} - 2\pi k)]$$

3. Relation between CT and DT Analyses

$$X_d(e^{j\omega_d}) = X_s\left[\left(\frac{\omega_d}{2\pi}\right) f_s\right]$$

$$X_s(f_a) = \frac{f_s}{2} \sum_k [\delta(f_a - f_{a0} - kf_s) + \delta(f_a + f_{a0} - kf_s)]$$

$$X_d(e^{j\omega_d}) = \frac{f_s}{2} \sum_k [\delta(\frac{\omega_d f_s}{2\pi} - f_{a0} - kf_s) + \delta(\frac{\omega_d f_s}{2\pi} + f_{a0} - kf_s)]$$

Recall $\delta(ax + b) = \frac{1}{|a|}\,\delta(x + b/a)$

$$\therefore \quad X_d(e^{j\omega_d}) = \frac{f_s}{2} \sum_k \left[\frac{2\pi}{f_s}\, \delta\left(\omega_d - \frac{2\pi f_{a0}}{f_s} - \frac{k2\pi f_s}{f_s}\right) \right.$$

$$\left. + \frac{2\pi}{f_s}\, \delta\left(\omega_d + \frac{2\pi f_{a0}}{f_s} - \frac{k2\pi f_s}{f_s}\right) \right]$$

$$= \pi \sum_k [\delta(\omega_d - \omega_{d0} - 2\pi k)$$

$$+ \delta(\omega_d + \omega_{d0} - 2\pi k)]$$

Aliasing

1. CT

$$f_{a1} = f_s/2 + \Delta_a$$

folds down to

$$f_{a2} = f_s/2 - \Delta_a$$

2. DT

Let $\omega_d = 2\pi \left(\frac{f_a}{f_s} \right) \qquad \Delta_d = 2\pi \left(\frac{\Delta_a}{f_s} \right)$

$$\omega_{d1} = \pi + \Delta_d$$

is identical to

$$\omega_{d2} = \pi - \Delta_d$$