The Life of a Cosine: Sampling, Digital Filtering, and Conversion Back to Analog

The purpose of this document is to trace a sinusoid at a single frequency through the block diagram shown below. We assume that the block labeled "DSP/Channel/Storage" consists only of a digital filter with frequency response $H^{\mathrm{d}}(\omega)$. We shall also assume that the signal is properly bandlimited. So there is no aliasing. There is, admittedly, a lot of material to wade through here. If you would rather get to the bottom line quickly, and skip the derivation, please note that by assumption, $x_1(t) = \cos\left(2\pi f_0 t\right)$, and look at (10) and (22).

The relationship (1) shown below that can be found on the Formula sheet for ECE 438 will be a critical element of the analysis. There is a proof of it in the Appendix at the end of this document.

$$\delta(ax+b) \equiv \left|\frac{1}{a}\right| \delta(x+b/a) \quad (1)$$

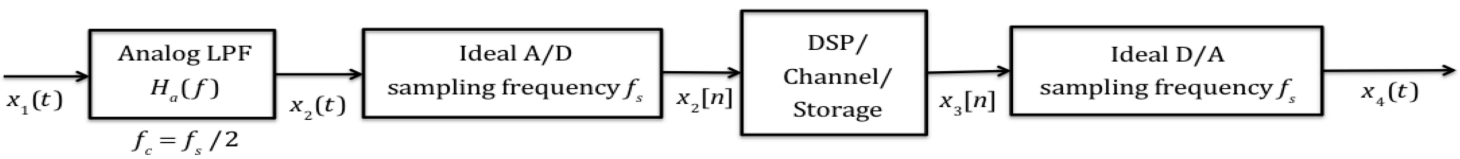

Fig. 1: Block diagram of processing system.

Let us now trace our way through this system, analyzing the signals at each stage. At the input, we assume that $x_1(t) = \cos\left(2\pi f_0 t\right)$. By assumption, we are sampling at f_s samples/sec, where f_0 satisfies the Nyquist condition. So $f_0 < f_c = f_s/2$; and $x_2(t) \equiv x_1(t)$.

We have that the Continuous-Time Fourier Transform (CTFT) of $x_2(t)$ is given by

$$X_2^{\mathrm{CTFT}}(f) = \frac{1}{2}\left\{\delta(f-f_0) + \delta(f+f_0)\right\}. \quad (2)$$

To determine the Discrete-Time Fourier Transform (DTFT) $X_2^{\mathrm{DTFT}}(\omega)$ of $x_2[n] = x_2(nT)$, where $T = 1/f_s$, we first define the continuous-time (CT) sampled version $x_{s2}(t)$ of $x_2(t)$, according to $x_2^s(t) = \mathrm{comb}_T\left[x_2(t)\right]$. Thus,

$$x_2^s(t) = \sum_{n=-\infty}^{\infty} x_2(nT)\delta(t-nT). \quad (3)$$

And

$$X_{s2}^{\text{CTFT}}(f) = f_s \text{rep}_{f_s}\left[X_2^{\text{CTFT}}(f)\right]. \tag{4}$$

or

$$X_{s2}^{\text{CTFT}}(f) = f_s \sum_{k=-\infty}^{\infty} X_2^{\text{CTFT}}(f - kf_s) = \frac{f_s}{2} \sum_{k=-\infty}^{\infty} \left\{\delta(f - f_0 - kf_s) + \delta(f + f_0 - kf_s)\right\}$$

We know that

$$X_2^{\text{DTFT}}(\omega) = X_{s2}^{\text{CTFT}}(f)\Big|_{f = f_s \frac{\omega}{2\pi}}. \tag{5}$$

Therefore,

$$X_2^{\text{DTFT}}(\omega) = \frac{f_s}{2} \sum_{k=-\infty}^{\infty} \left\{\delta\left(f_s \frac{\omega}{2\pi} - f_0 - kf_s\right) + \delta\left(f_s \frac{\omega}{2\pi} + f_0 - kf_s\right)\right\}. \tag{6}$$

Using the identity (1) with $a = f_s / 2\pi$ in (6), we obtain

$$\begin{aligned} X_2^{\text{DTFT}}(\omega) &= \pi \sum_{k=-\infty}^{\infty} \left\{\delta\left(\omega - 2\pi(f_0 - kf_s)/f_s\right) + \delta\left(\omega - 2\pi(-f_0 - kf_s)/f_s\right)\right\} \\ &= \pi\left\{\delta\left(\omega - 2\pi f_0/f_s\right) + \delta\left(\omega + 2\pi f_0/f_s\right)\right\}, 0 \le |\omega| \le \pi \qquad , \\ &= \pi\left\{\delta\left(\omega - \omega_0\right) + \delta\left(\omega + \omega_0\right)\right\}, 0 \le |\omega| \le \pi \end{aligned} \tag{7}$$

where $\omega_0 = 2\pi f_0 / f_s$. Thus,

$$x_3[n] = \cos(\omega_0 n). \tag{8}$$

Whew! All that work to just get this simple result!

Now we apply a digital filter with frequency response $H^{\text{d}}(\omega)$ to $x_3[n]$. Then, referring again to the block diagram in Fig. 1, the DTFT $X_3^{\text{DTFT}}(\omega)$ of the filter output $x_3[n]$ is given by $X_3^{\text{DTFT}}(\omega) = H^{\text{d}}(\omega) X_2^{\text{DTFT}}(\omega)$, which based on the last line of (7), is given by

$$X_3^{\text{DTFT}}(\omega) = \pi\left\{H^{\text{d}}(\omega_0)\delta\left(\omega - \omega_0\right) + H^{\text{d}}(-\omega_0)\delta\left(\omega + \omega_0\right)\right\}, 0 \le |\omega| \le \pi. \tag{9}$$

Here, we have used the identity $g(x)\delta(x-x_0)\equiv g(x_0)\delta(x-x_0)$, assuming that $g(x)$ is continuous at $x=x_0$. Next, using the fact that $H^{\mathrm{d}}(\omega)=\left|H^{\mathrm{d}}(\omega)\right|e^{j\underline{/H^{\mathrm{d}}(\omega)}}$, the symmetry $H^{\mathrm{d}}(-\omega)=H^{\mathrm{d}}{}^{*}(\omega)$, which follows from the assumption that any real-valued input to the system yields a real-valued output, and Euler's identity for the cosine, we have that the filter output $x_3[n]$ can be expressed as

$$x_3[n]=\left|H^{\mathrm{d}}(\omega_0)\right|\cos\left(\omega_0 n+\underline{/H^{\mathrm{d}}(\omega_0)}\right). \tag{10}$$

Now, we are ready to go back to the continuous-time domain. Referring again to Fig. 1, we can model the ideal D/A convertor by first considering the continuous-time version $x_{s4}(t)$ of the DT signal $x_4[n]$ given by

$$x_{s4}(t)=\sum_{n=-\infty}^{\infty}x_4[n]\delta(t-nT). \tag{11}$$

Note here that $x_4[n]\equiv x_3[n]$. This signal is passed through an idea low-pass filter

$$H_{\mathrm{ILP}}^{\mathrm{a}}(f)=T\,\mathrm{rect(Tf)}, \tag{12}$$

with gain T. Here the superscript "a" denotes "analog" to distinguish this filter from a digital filter denoted by superscript "d".

In the time domain, we have a convolution with the impulse response $\mathrm{h}_{\mathrm{ILP}}^{\mathrm{a}}(t)=\mathrm{sinc(t/T)}$ of this filter, which results in a replication of the impulse response where each impulse is located in $x_{s4}(t)$. This will properly interpolate between the samples $x_4[n]=x_4(nT)$ of the DT input to the ideal D/A convertor. So we have

$$x_4(t)=\sum_{n=-\infty}^{\infty}x_4(nT)\mathrm{sinc}\left((t-nT)/T\right). \tag{13}$$

Now, let's see how this all works out in the frequency domain. We have

$$X_{s4}^{\mathrm{CTFT}}(f)=X_3^{DTFT}(\omega)\Big|_{\omega=2\pi\frac{f}{f_s}}, \tag{14}$$

which is effectively the inverse of (5). Also,

$$\begin{aligned} X_{s4}^{\mathrm{CTFT}}(f) &= f_s \operatorname{rep}_{f_s}\left[X_4(f)\right] \\ &= f_s \sum_{k=-\infty}^{\infty} X_4(f-kf_s) \end{aligned} \tag{15}$$

Now, from (9), we have

$$X_3^{\mathrm{DTFT}}(\omega) = \pi\left\{H^{\mathrm{d}}(\omega_0)\delta\left(\omega-\omega_0\right)+H^{\mathrm{d}}(-\omega_0)\delta\left(\omega+\omega_0\right)\right\}, 0 \le |\omega| \le \pi\ . \tag{16}$$

Expanding this to make it periodic and remove the domain restriction, we obtain

$$X_3^{\mathrm{DTFT}}(\omega) = \pi \sum_{k=-\infty}^{\infty} H^{\mathrm{d}}(\omega_0)\delta\left(\omega-\omega_0-2\pi k\right)+H^{\mathrm{d}}(-\omega_0)\delta\left(\omega+\omega_0-2\pi k\right). \tag{17}$$

Then, applying (14) to (17), yields

$$X_{s4}^{\mathrm{CTFT}}(f) = \pi \sum_{k=-\infty}^{\infty} H^{\mathrm{d}}(\omega_0)\delta\left(2\pi\frac{f}{f_s}-\omega_0-2\pi k\right)+H^{\mathrm{d}}(-\omega_0)\delta\left(2\pi\frac{f}{f_s}+\omega_0-2\pi k\right). \tag{18}$$

Next, we again use the identity (1), but this time with $a = 2\pi / f_s$ to get

$$X_{s4}^{\mathrm{CTFT}}(f) = \frac{f_s}{2} \sum_{k=-\infty}^{\infty} H^{\mathrm{d}}(\omega_0)\delta\left(f-\frac{f_s}{2\pi}(\omega_0-2\pi k)\right)+H^{\mathrm{d}}(-\omega_0)\delta\left(f-\frac{f_s}{2\pi}(-\omega_0+2\pi k)\right). \tag{19}$$

We also note that $\omega_0 = 2\pi f_0 / f_s$. After using this in (19), we have

$$X_{s4}^{\mathrm{CTFT}}(f) = \frac{f_s}{2} \sum_{k=-\infty}^{\infty} H^{\mathrm{d}}(2\pi f_0 / f_s)\delta\left(f-f_0-kf_s\right)+H^{\mathrm{d}}(-2\pi f_0 / f_s)\delta\left(f+f_0-kf_s\right). \tag{20}$$

Applying the low-pass filter (12) to (20), blocks all but the $k = 0$ in (20), under our assumption that $f_0 < f_s / 2$, i.e. we are satisfying the Nyquist sampling condition. Also, the filter gain T in (12) cancels the factor f_s immediately to the right of the equal sign in (20). So we finally have

$$\begin{aligned} X_4^{\mathrm{CTFT}}(f) &= H_{\mathrm{ILP}}^{\mathrm{a}}(f) X_{s4}^{\mathrm{CTFT}}(f) \\ &= \frac{1}{2}\left\{H^{\mathrm{d}}(2\pi f_0 / f_s)\delta\left(f-f_0\right)+H^{\mathrm{d}}(-2\pi f_0 / f_s)\delta\left(f+f_0\right)\right\}. \end{aligned} \tag{21}$$

Transforming this expression to the time domain, we obtain

$$x_4(t) = \left|H^{\mathrm{d}}(2\pi f_0 / f_s)\right| \cos\left(2\pi f_0 t + \underline{/H^{\mathrm{d}}(2\pi f_0 / f_s)}\right), \tag{22}$$

which is our final result!!

Appendix A
Proof of Equation (1)

Here we prove (1), which for convenience is restated here as (A1):

$$\delta(ax+b) \equiv \left|\frac{1}{a}\right| \delta(x+b/a) \tag{A1}$$

There are many ways to obtain the ideal CT impulse function $\delta(t)$ as a limiting case of real signals. One example is

$$\delta(t) = \lim_{\Delta \to 0} \frac{1}{\Delta} \mathrm{rect}(t/\Delta). \tag{A2}$$

Using any one of these forms, we can prove the two critical identities (A3) and (A4) for the impulse function:

$$g(x_0) = \int_{-\infty}^{\infty} g(\xi)\delta(\xi - x_0)\mathrm{d}\xi, \tag{A3}$$

and

$$\begin{aligned} g(x-x_0) &= g(x) * \delta(x-x_0) \\ &= \int_{-\infty}^{\infty} g(\xi)\delta(x - x_0 - \xi)\mathrm{d}\xi \end{aligned} . \tag{A4}$$

The third critical identity (A5) for the impulse function follows from (A3)

$$g(x)\delta(x-x_0) \equiv g(x_0)\delta(x-x_0). \tag{A5}$$

All five of the identities (A1)-(A5) can be found on the Formula sheet for ECE 438. Here, for these identities to be valid, it is assumed that $g(x)$ is continuous at $x = x_0$. It is possible to derive special cases for them when this assumption is not satisfied. It is also possible to derive special cases when the limits of integration in (A3) are not infinite. However, these special cases are beyond the scope of what we wish to consider here.

To prove (A1), we will simply show that (A3) is satisfied under the assumption that (A1) holds. In particular, let us assume that $a > 0$, and consider

$$\int_{-\infty}^{\infty} g(\xi)\delta(a\xi - x_0)\mathrm{d}\xi = \frac{1}{|a|}\int_{-\infty}^{\infty} g(\eta / a)\delta(\eta - x_0)\mathrm{d}\eta\,. \tag{A5}$$

Here we have made a change of the variable of integration $\eta = a\xi$ in (A3) to obtain the expression on the right side of the equal sign. Note that we have made the substitution $\mathrm{d}\xi = \mathrm{d}\eta / a$. For $a < 0$, we end up reversing the limits of integration, But they can be restored to go from $-\infty$ to ∞ by putting a minus sign in front of the term $1/a$. Then since $a < 0$ in this case, we can replace $1/a$ by $1/|a|$, and the expression will be valid for both $a > 0$ and $a < 0$. Now, by (A2), we have that the right side of (A5) can be expressed as

$$\frac{1}{|a|}\int_{-\infty}^{\infty} g(\eta / a)\delta(\eta + b)\mathrm{d}\eta = \frac{1}{|a|} g(-b / a)\,. \tag{A6}$$

But according to (A2), we also have that

$$\int_{-\infty}^{\infty} g(\eta)\frac{1}{|a|}\delta(\eta + b / a)\mathrm{d}\eta = \frac{1}{|a|} g(-b / a)\,. \tag{A7}$$

Thus, from the left side of (A5) and both sides of (A6) and (A7), we have that

$$\delta(ax + b) \equiv \left|\frac{1}{a}\right|\delta(x + b / a)\,,$$

which is what we wanted to prove. The proof that (A4) holds follows similarly.