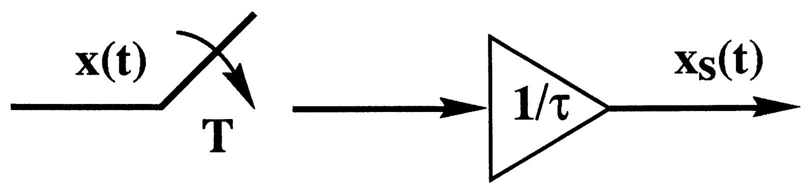
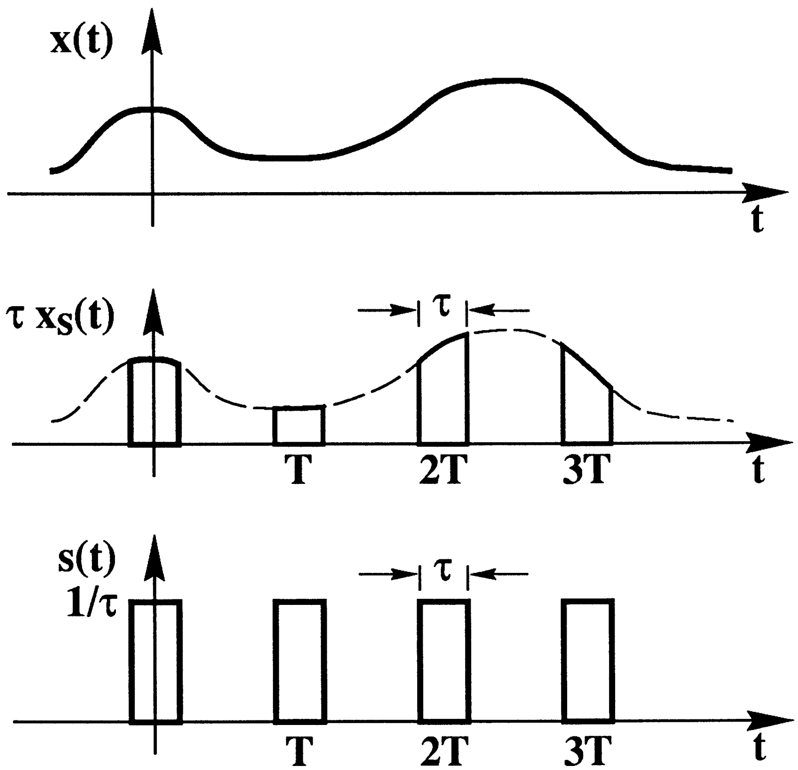
1.4.1 ANALYSIS OF SAMPLING

A simple scheme for sampling a waveform is to *gate* it.



T	—	period
τ	—	interval for which switch is closed
τ/T	—	duty cycle



$x_S(t) = s(t)\, x(t)$

Fourier Analysis

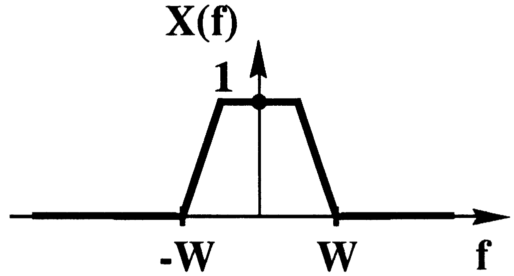
$$X_s(f) = S(f) * X(f)$$

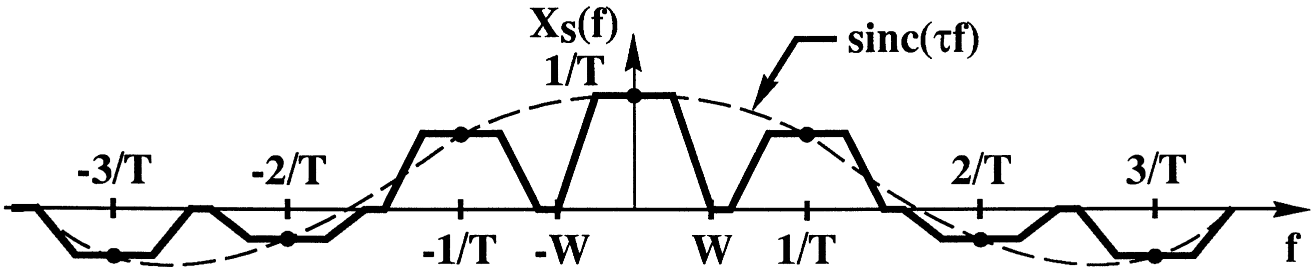
$$s(t) = \text{rep}_T[\frac{1}{\tau} \text{ rect } (\frac{t}{\tau})]$$

$$S(f) = \frac{1}{T} \text{ comb}_{\frac{1}{T}} \text{ [sinc } (\tau f)]$$

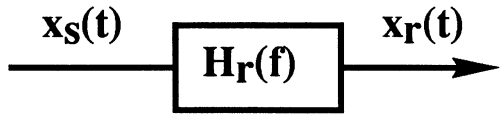
$$= \frac{1}{T} \sum_k \text{ sinc } (\tau k/T) \; \delta(f - k/T)$$

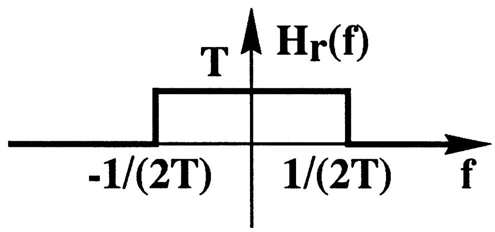
$$X_s(f) = \frac{1}{T} \sum_k \text{ sinc } (\tau k/T) \; X(f - k/T)$$





How do we reconstruct x(t) from $x_s(t)$?





Nyquist condition

Perfect reconstruction of $x(t)$ from $x_s(t)$ is possible if $X(f) = 0, \ |f| \geq 1/(2T)$

Nyquist sampling rate

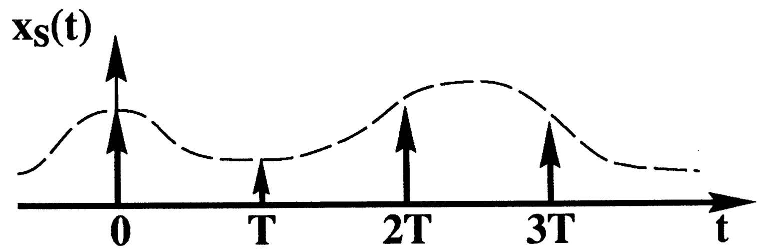
$$f_s = \frac{1}{T} = 2W$$

Ideal Sampling

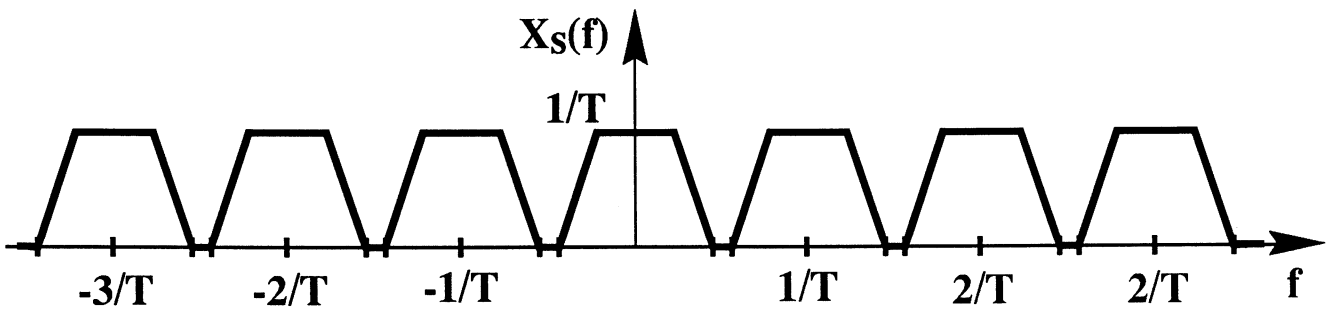
What happens as $\tau \rightarrow 0$?

$$s(t) \rightarrow \sum_m \delta(t - mT)$$

$$x_s(t) \rightarrow \sum_m x(mT)\, \delta(t - mT) = \mathrm{comb}_T[x(t)]$$



$$X_s(f) \rightarrow \frac{1}{T} \sum_k X(f - k/T) = \frac{1}{T} \mathrm{rep}_{\frac{1}{T}} [X(f)]$$



- An ideal lowpass filter will again reconstruct x(t).

- In the sequel, we assume ideal sampling.

Transform Relations

$$\mathrm{rep}_T[x(t)] \overset{\text{CTFT}}{\longleftrightarrow} \frac{1}{T}\,\mathrm{comb}_{\frac{1}{T}}[X(f)]$$

$$\mathrm{comb}_T[x(t)] \overset{\text{CTFT}}{\longleftrightarrow} \frac{1}{T}\,\mathrm{rep}_{\frac{1}{T}}[X(f)]$$

Given one relation, the other follows by reciprocity.

Whittaker-Kotelnikov-Shannon Sampling Expansion

$$X_r(f) = H_r(f)\, X_s(f)$$

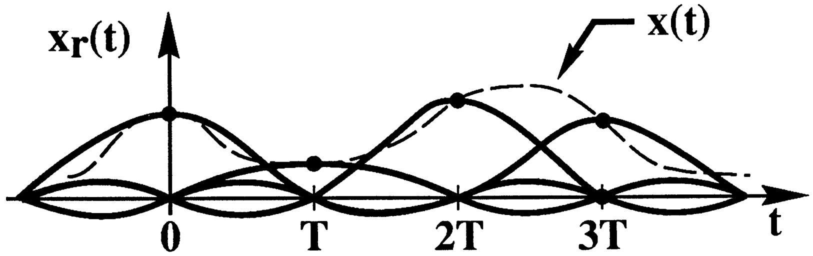
$$H_r(f) = T \text{ rect } (Tf)$$

$$x_r(t) = h_r(t) * x_s(t)$$

$$h_r(t) = \text{sinc } (t/T)$$

$$x_s(t) = \sum_m x(mT)\, \delta(t - mT)$$

$$x_r(t) = \sum_m x(mT)\,\mathrm{sinc}\left(\frac{t - mT}{T}\right)$$

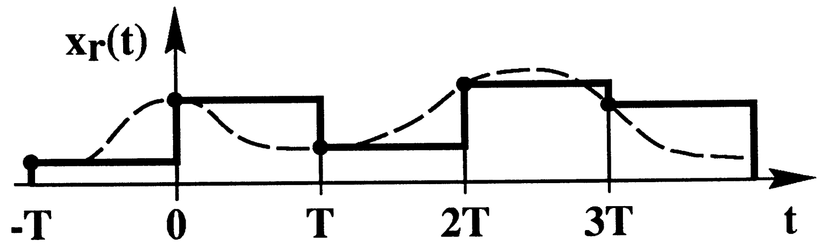


$$x_r(nT) = \sum_m x(mT)\,\mathrm{sinc}\,(n - m)$$

$$= x(nT)$$

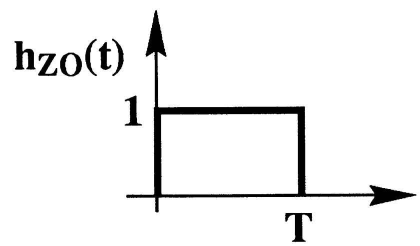
If Nyquist condition is satisfied, $x_r(t) \equiv x(t)$.

Zero Order Hold Reconstruction



$$x_r(t) = \sum_m x(mT)\,\text{rect}\left(\frac{t - T/2 - mT}{T}\right)$$

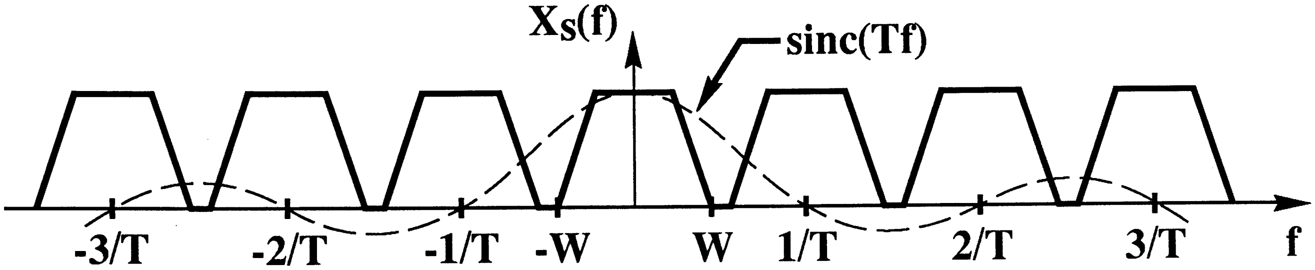
$$= h_{ZO}(t) * x_s(t)$$

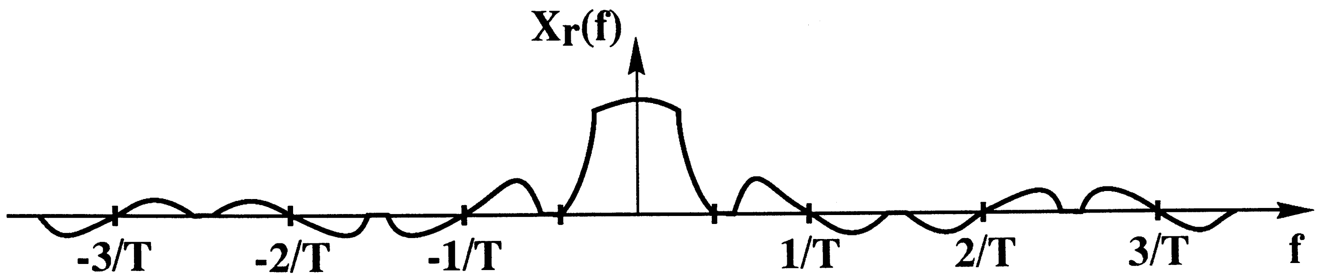


$$h_r(t) = \text{rect}\left(\frac{t - T/2}{T}\right)$$

$$X_r(f) = H_{ZO}(f)\, X_s(f)$$

$$H_{ZO}(f) = T \text{ sinc } (Tf)\, e^{-j2\pi f(T/2)}$$

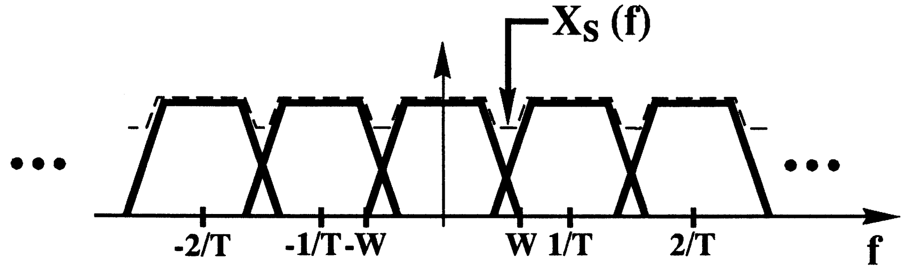




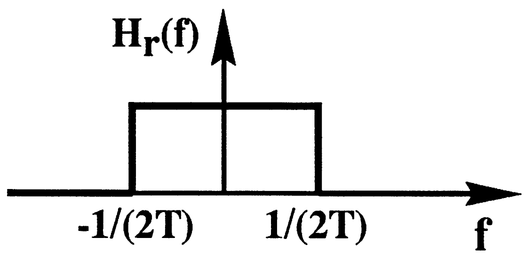
- One way to overcome the limitations of zero order hold reconstruction is to oversample, *i.e.* choose $f_s = 1/T \gg W$.

- This moves spectral replications farther out to where they are better attenuated by sinc(Tf).

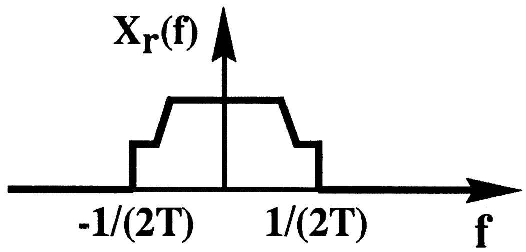
- What happens if we inadvertently undersample, and then reconstruct with an ideal lowpass filter?



X_S (f)
-2/T
-1/T
-W
W
1/T
2/T
f

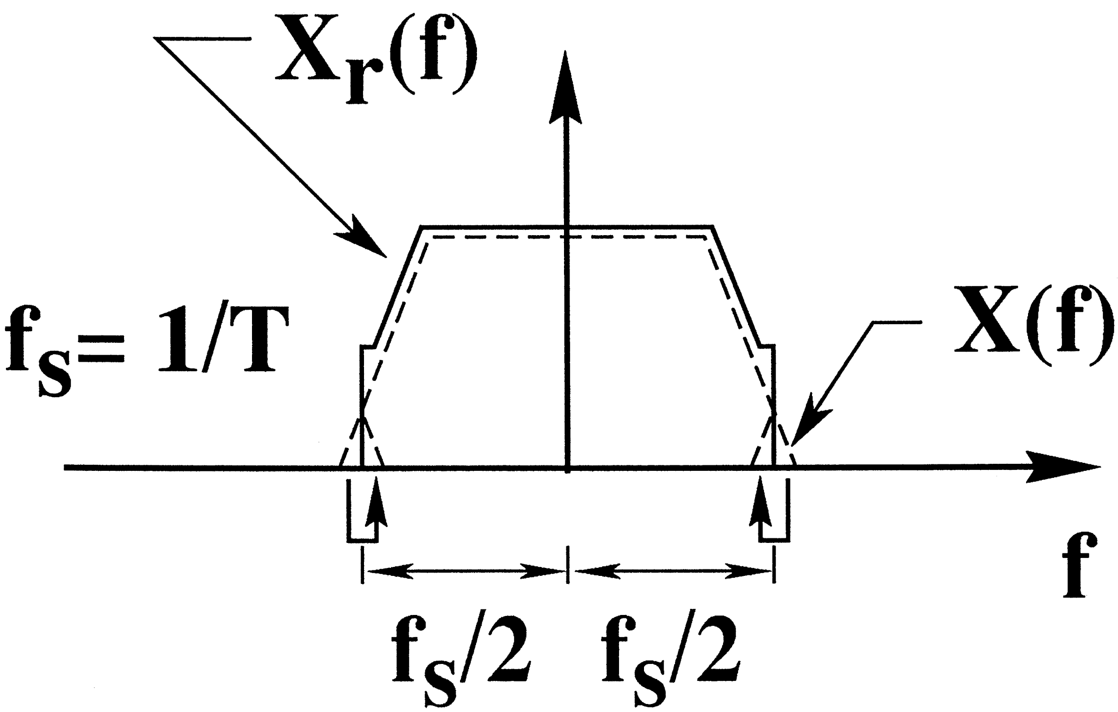


H_r(f)
-1/(2T)
1/(2T)
f



X_r(f)
-1/(2T)
1/(2T)
f

Effect of Undersampling



- frequency truncation error

$$X_r(f) = 0 \,, \qquad |f| \geq f_s/2$$

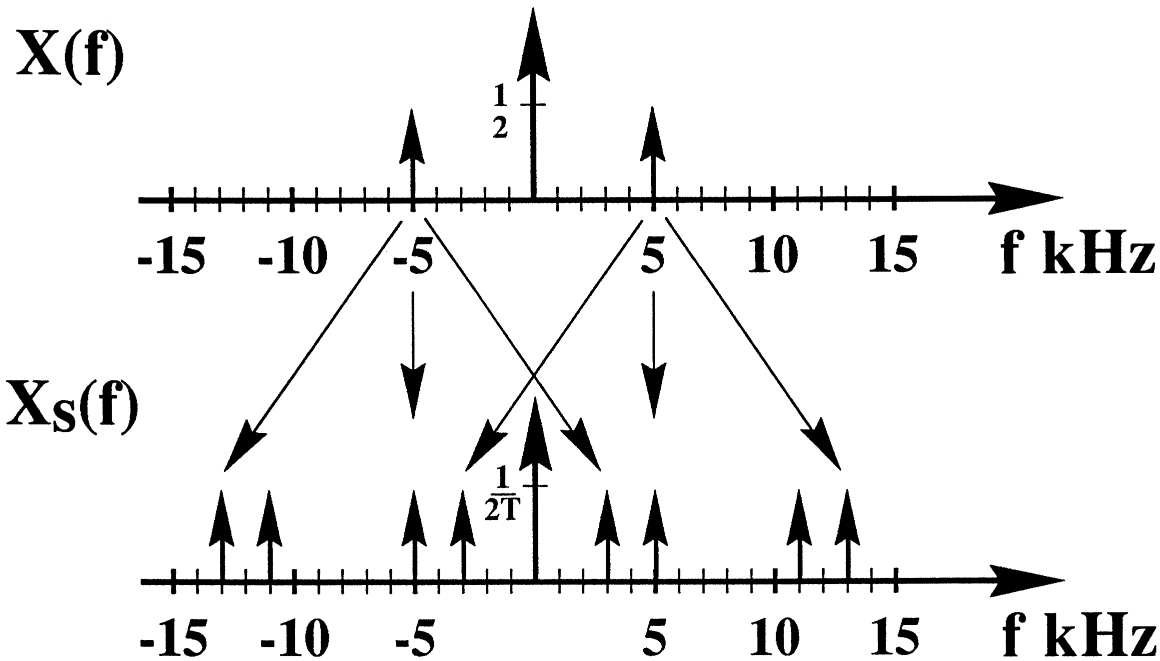
- aliasing error

  frequency components at $f_1 = f_s/2 + \Delta$ fold down to and mimic frequencies $f_2 = f_s/2 - \Delta$.

Example

- $x(t) = \cos[2\pi(5000)t]$
- sample at $f_s = 8$ kHz
- reconstruct with ideal lowpass filter having cutoff at 4 kHz.

Sampling



$$x_r(t) = \cos[2\pi(3000)t]$$

Note that x(t) and $x_r(t)$ will have the same sample values at times $t = nT \quad (T = 1/(8000))$

$$x(nT) = \cos\{2\pi(5000)n/8000\}$$

$$= \cos\{2\pi[(5000)n - (8000)n]/8000\}$$

$$= \cos\{-2\pi(3000)n/8000\}$$

$$= x_r(nT)$$

Reconstruction