1.3.3 DISCRETE-TIME FOURIER TRANSFORM (DTFT)

- Spectral representation for *aperiodic* DT signals.

- As in the CT case, we may derive the DTFT by starting with a spectral representation (the discrete-time Fourier series) for periodic DT signals and letting the period become infinitely long.

- Instead, we will take a shorter but less direct approach.

Recall the continuous-time Fourier series:

$$x(t) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_k e^{j2\pi kt/T} \tag{1}$$

$$X_k = \int_{-T/2}^{T/2} x(t)\, e^{-j2\pi kt/T} dt \tag{2}$$

Before, we interpreted (1) as a spectral representation for the CT periodic signal x(t).

Now, let's consider (2) to be a spectral representation for the sequence X_k, $-\infty < k < \infty$.

We are effectively interchanging the time and frequency domains.

We want to express an arbitrary signal x(n) in terms of complex exponential signals of the form $e^{j\omega n}$.

Recall that ω is only unique modulo 2π.

To obtain this, we make the following substitutions in (2)

$$X_k = \int_{-T/2}^{T/2} x(t)\, e^{-j2\pi kt/T} dt$$

$$k \rightarrow n \qquad Tx(t) \rightarrow X(e^{j\omega}) \qquad -2\pi t/T \rightarrow \omega$$

$$X_k \rightarrow x(n) \qquad\qquad dt \rightarrow -\frac{T}{2\pi}\, d\omega$$

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega$$

This is the inverse transform.

To obtain the forward transform, we make the same substitutions in (1)

$$x(t) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_k \, e^{j2\pi kt/T}$$

$$Tx(t) \rightarrow X(e^{j\omega}) \qquad k \rightarrow n \qquad 2\pi t/T \rightarrow -\omega$$

$$X_k \rightarrow x(n)$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) \, e^{-j\omega n}$$

Putting everything together

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega$$

Sufficient conditions for existence

$$\sum_n |\, x(n)\, |^2 < \infty$$

or

$$\sum_n |\, x(n)\, | < \infty$$ plus Dirichlet conditions.

Transform Relations

1. linearity

$$a_1 x_1(n) + a_2 x_2(n) \overset{\text{DTFT}}{\longleftrightarrow} a_1 X_1(e^{j\omega}) + a_2 X_2(e^{j\omega})$$

2. shifting

$$x(n - n_0) \overset{\text{DTFT}}{\longleftrightarrow} X(e^{j\omega})\, e^{-j\omega n_0}$$

3. modulation

$$x(n)\, e^{j\omega_0 n} \overset{\text{DTFT}}{\longleftrightarrow} X(e^{j(\omega-\omega_0)})$$

4. Parseval's relation

$$\sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega$$

5. Initial value

$$\sum_{n=-\infty}^{\infty} x(n) = X(e^{j0})$$

Comments

1. As discussed earlier, scaling in DT involves sampling rate changes; thus, the transform relations are more complicated than in CT case.

2. There is no reciprocity relation, because time domain is discrete-parameter whereas frequency domain is continuous-parameter.

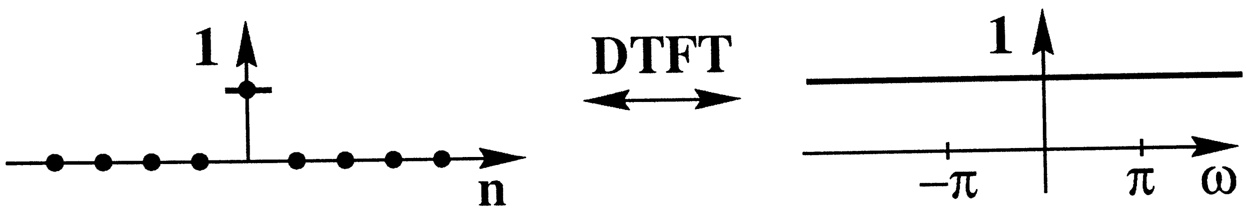
3. As in CT case, Parseval's relation guarantees uniqueness of the DTFT.

Some Transform Pairs

1. $x(n) = \delta(n)$

$$X(e^{j\omega}) = \sum_{n} \delta(n)\, e^{-j\omega n}$$

$$= 1 \quad \text{(by sifting property)}$$



2. $x(n) = 1$

- does not satisfy existence conditions
- $\sum_n e^{-j\omega n}$ does not converge in ordinary sense
- cannot use reciprocity as we did for CT case.

Consider inverse transform

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n} d\omega$$

What function $X(e^{j\omega})$ would yield $x(n) \equiv 1$?

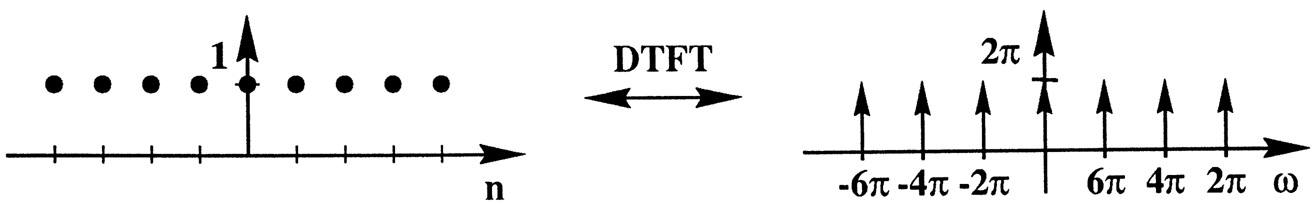
Let $X(e^{j\omega}) = 2\pi\ \delta(\omega),\ -\pi \leq\ \omega \leq\ \pi$

then $\quad x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} 2\pi\ \delta(\omega)\ e^{j\omega n} d\omega$

$= 1$ (again by sifting property)

Note that $X(e^{j\omega})$ must be periodic with period 2π; so we have

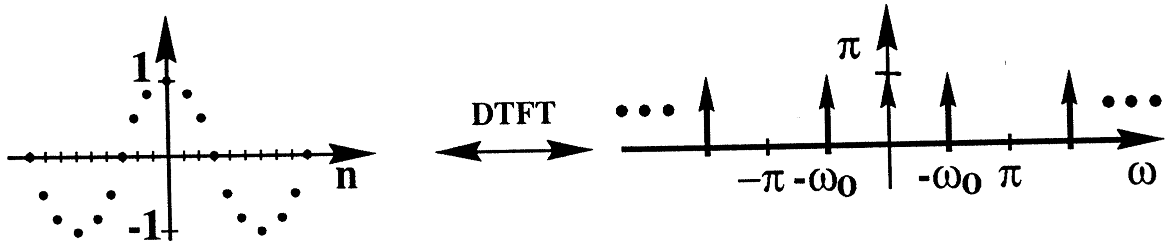
$$X(e^{j\omega}) = 2\pi \sum_{k} \delta(\omega - 2\pi k)$$



3. $e^{j\omega_0 n} \overset{\text{DTFT}}{\longleftrightarrow} \text{rep}_{2\pi}[2\pi\ \delta(\omega - \omega_0)]$

(by modulation property)

4. $\cos(\omega_0 n) \overset{\text{DTFT}}{\longleftrightarrow} \text{rep}_{2\pi}[\pi\ \delta(\omega - \omega_0) + \pi\ \delta(\omega + \omega_0)]$



5. $$x(n) = \begin{cases} 1\,, & 0 \le n \le N-1 \\ 0\,, & \text{else} \end{cases}$$

$$X(e^{j\omega}) = \sum_{n=0}^{N-1} e^{-j\omega n}$$

$$= \frac{1 - e^{-j\omega N}}{1 - e^{-j\omega}}$$

$$= e^{-j\omega(N-1)/2} \; \frac{\sin(\omega N/2)}{\sin(\omega/2)}$$

$$6.\ y(n) = \begin{cases} 1, & -(N-1)/2 \le n \le (N-1)/2 \\ 0, & \text{else} \end{cases} \qquad N \text{ odd}$$

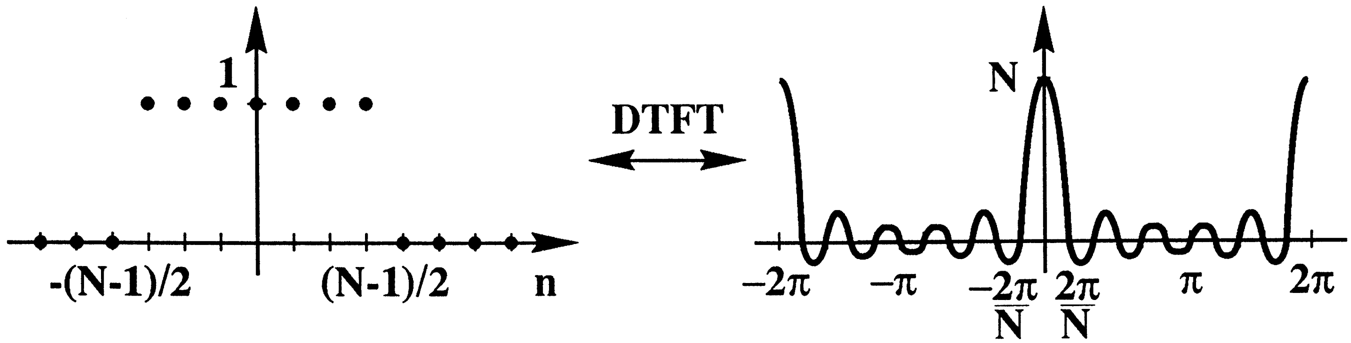
$$y(n) = x(n + (N-1)/2)$$

where x(n) is signal from Example 5

$$Y(e^{j\omega}) = X(e^{j\omega})\, e^{j\omega(N-1)/2} \quad \text{(by shifting property)}$$

$$= \left[e^{-j\omega(N-1)/2} \frac{\sin(\omega N/2)}{\sin(\omega/2)} \right] e^{j\omega(N-1)/2}$$

$$Y(e^{j\omega}) = \frac{\sin(\omega N/2)}{\sin(\omega/2)}$$



What happens as $N \to \infty$?

$y(n) \to 1\ ,\ -\infty < n < \infty \qquad 2\pi/N \to 0$

$Y(e^{j\omega}) \to \mathrm{rep}_{2\pi}[2\pi\ \delta(\omega)] \qquad \frac{1}{2\pi} \int_{-\pi}^{\pi} Y(e^{j\omega})\, d\omega = y(0)$

DTFT and DT LTI Systems

Recall that in CT case we obtained a general characterization in terms of CTFT by expressing x(t) as a superposition of complex exponential signals and then using frequency response H(f) to determine response to each such signal.

Here we take a different approach.

For any DT LTI system, we know that input and output are related by

$$y(n) = \sum_{k} h(n-k)\, x(k)$$

Thus

$$Y(e^{j\omega}) = \sum_{n} y(n)\, e^{-j\omega n}$$

$$= \sum_{n} \sum_{k} h(n-k)\, x(k)\, e^{-j\omega n}$$

$$= \sum_{k} \{ \sum_{n} h(n-k)\, e^{-j\omega n} \}\, x(k)$$

$$= \sum_{k} H(e^{j\omega})\, e^{-j\omega k}\, x(k) \quad \text{(by shifting property)}$$

where $H(e^{j\omega})$ is the DTFT of the impulse response $h(n)$.

Rearranging,

$$Y(e^{j\omega}) = H(e^{j\omega}) \sum_k x(k)\, e^{-j\omega k}$$

$$= H(e^{j\omega})\, X(e^{j\omega})$$

How is $H(e^{j\omega})$ related to the frequency response?

Consider

$$x(n) = e^{j\omega_0 n}$$

$$X(e^{j\omega}) = \text{rep}_{2\pi}[2\pi\ \delta(\omega - \omega_0)]$$

$$X(e^{j\omega}) = 2\pi \sum_{k} \delta(\omega - \omega_0 - 2\pi k)$$

$$Y(e^{j\omega}) = H(e^{j\omega})\, X(e^{j\omega})$$

$$= 2\pi \sum_{k} H(e^{j\omega_0 + 2\pi k})\, \delta(\omega - \omega_0 - 2\pi k)$$

$$= H(e^{j\omega_0})\, 2\pi \sum_{k} \delta(\omega - \omega_0 - 2\pi k)$$

$$Y(e^{j\omega}) = H(e^{j\omega_0})\, X(e^{j\omega})$$

$$\therefore \quad y(n) = H(e^{j\omega_0})\, x(n)$$

so $H(e^{j\omega})$ is also the frequency response of the DT system.

We thus have two equivalent characterizations for the response y(n) of a DT LTI system to any input x(n)

$$y(n) = \sum_{k} h(n-k)\, x(k)$$

$$Y(e^{j\omega}) = H(e^{j\omega})\, X(e^{j\omega})$$

Convolution Theorem

Since x(n) and h(n) are arbitrary signals, we also have the following transform relation

$$\sum_{k} x_1(k)\, x_2(n-k) \overset{\text{DTFT}}{\longleftrightarrow} X_1(e^{j\omega})\, X_2(e^{j\omega})$$

or

$$x_1(n) * x_2(n) \overset{\text{DTFT}}{\longleftrightarrow} X_1(e^{j\omega})\, X_2(e^{j\omega})$$

Product Theorem

As mentioned earlier, we do not have a reciprocity relation for the DT case.

However, by direct evaluation of the DTFT, we also have the following result

$$x_1(n)\, x_2(n) \overset{\text{DTFT}}{\longleftrightarrow} \frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(e^{j(\omega-\mu)})\, X_2(e^{j\mu})\, d\mu$$

Note that this is periodic convolution.