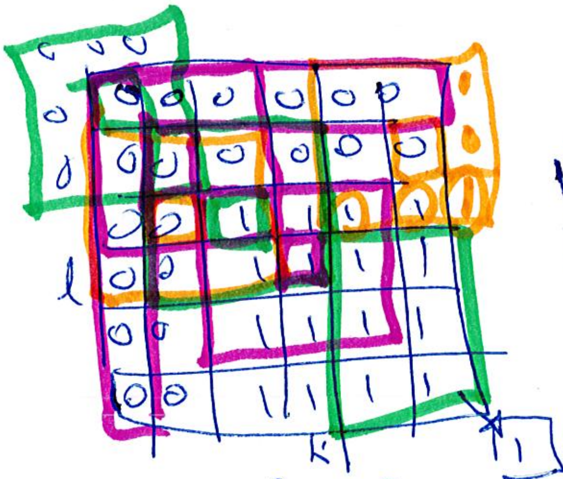
Spatial filtering

$$g[m,n] = \sum_k \sum_l h[-(k-m), -(l-n)]\, f[k,l]$$

Example 1



f[k,l]

1/16	1/8	1/16
1/8	1/4	1/8
1/16	1/8	1/16

h[k,l]

0	0	0	0	0	0
0	1/16	3/16	4/16	4/16	4/16
0	3/16	9/16	12/16	12/16	12/16
0	4/16	12/16	1	1	1
0	4/16	12/16	1	1	1
0	4/16	12/16	1	1	1

g[m,n]

Filter is DC-preserving

$$\Longleftrightarrow \sum_k \sum_l h[k,l] = 1$$

blurring or smoothing

Frequency Domain Analysis

Definition DSFT (Discrete-space Fourier transform)

Forward
$$F(\mu, \nu) = \sum_m \sum_n f[m,n]\, e^{-j(m\mu + n\nu)}.$$

μ ↔ m, ν ↔ n

(μ, ν) – radians/sample

Inverse
$$f[m,n] = \left(\frac{1}{2\pi}\right)^2 \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} F(\mu,\nu)\, e^{j(m\mu + n\nu)}\, d\mu\, d\nu$$

Analysis of Filtering

$$g[m,n] = \sum_k \sum_l h[m-k, n-l]\, f[k,l]$$

$$G(\mu,\nu) = H(\mu,\nu)\, F(\mu,\nu)$$

2D discrete - convolution theorem

Application to Example 1

$h[k,l]$

1/16	1/8	1/16
1/8	1/4	1/8
1/16	1/8	1/16

$l=0$

1/4
1/2
1/4

$h[l]$ $\Rightarrow h[k,l] = h_1[k]\, h_1[l]$

$k=0$

1/4	1/2	1/4

$h_1[k]$

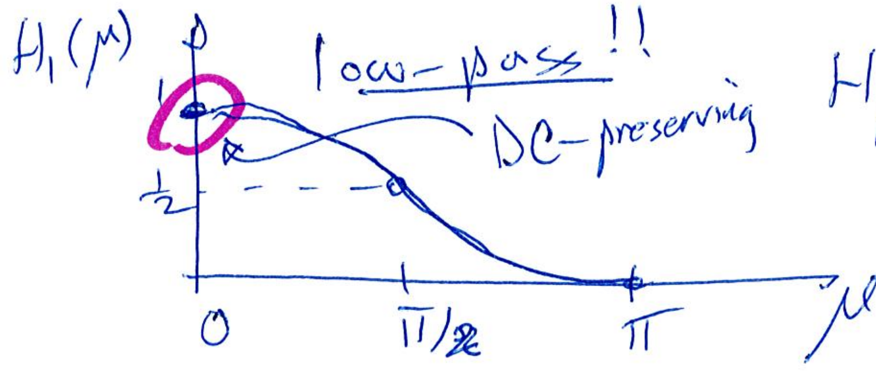
By separability,

$$H(\mu,\nu) = H_1(\mu)\, H_1(\nu)$$

$$H_1(\mu) = \sum_k h_1[k]\, e^{-jk\mu}$$

$$= \frac{1}{4} e^{j\mu} + \frac{1}{2} e^{j0} + \frac{1}{4} e^{-j\mu}$$

$$= \frac{1}{2} + \frac{1}{2}\cos(\mu)$$



$$H(\mu,\nu) = \left[\frac{1}{2} + \frac{1}{2}\cos(\mu)\right] \times \left[\frac{1}{2} + \frac{1}{2}\cos(\nu)\right]$$

Example 2

$$f[m,n] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix} \qquad h[k,l] = \begin{bmatrix} -1/9 & -1/9 & -1/9 \\ -1/9 & 8/9 & -1/9 \\ -1/9 & -1/9 & -1/9 \end{bmatrix}$$

$8/9$

$$g[m,n] = \begin{bmatrix} \boxed{0} & 0 & 0 & 0 & 0 & 0 \\ 0 & \boxed{-1/9} & -2/9 & -3/9 & -3/9 & -3/9 \\ 0 & -2/9 & \boxed{5/9} & 3/9 & 3/9 & 3/9 \\ 0 & -3/9 & 3/9 & \boxed{0} & 0 & 0 \\ 0 & -3/9 & 3/9 & 0 & 0 & 0 \\ 0 & -3/9 & 3/9 & 0 & 0 & 0 \end{bmatrix}$$

Edge detector – only responds to changes in input $\Rightarrow \sum_k \sum_l h[k,l] = 0$

Well-known examples

Sobel edge detector

Canny edge detector

Frequency response

$h[m,n] = h_1[m,n] - h_2[m,n]$

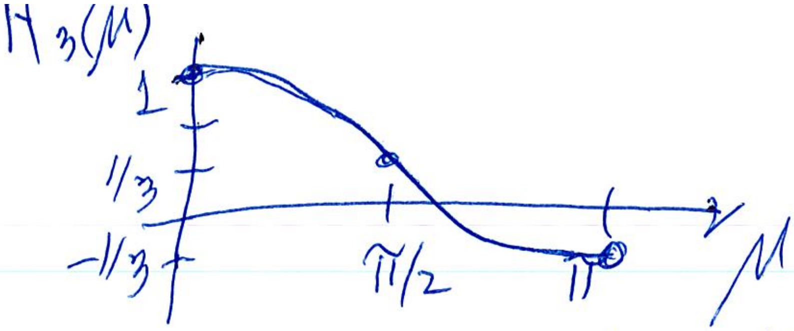
$$\begin{bmatrix} -1/9 & -1/9 & -1/9 \\ -1/9 & 8/9 & -1/9 \\ -1/9 & -1/9 & -1/9 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 1/9 & 1/9 & 1/9 \\ 1/9 & 1/9 & 1/9 \\ 1/9 & 1/9 & 1/9 \end{bmatrix}$$

$$h_3[n] = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix} \qquad h_3[m] = \begin{bmatrix} 1/3 & 1/3 & 1/3 \end{bmatrix}$$

$h_2[m,n] = h_3[m]\, h_3[n]$

$$H_3(\mu) = \frac{1}{3}e^{j\mu} + \frac{1}{3} + \frac{1}{3}e^{-j\mu}$$

$$= \frac{1}{3}\left[1 + 2\cos(\mu)\right]$$

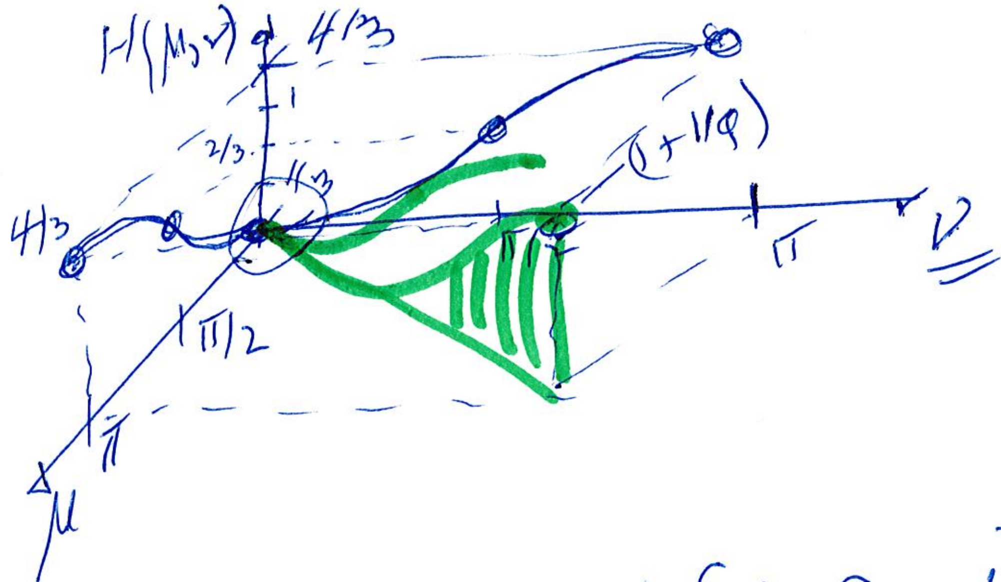


$$H(\mu,\nu) = H_1(\mu,\nu) - H_2(\mu,\nu)$$

$$H_1(\mu,\nu) \equiv 1 \,, \quad H_2(\mu,\nu) = H_3(\mu) H_3(\nu)$$

$$H(\mu,\nu) = 1 - \frac{1}{9}\left[1 + 2\cos\mu\right]\left[1 + 2\cos\nu\right]$$

$$H(\mu,0) = 1 - \frac{1}{3}\left[1 + 2\cos\mu\right]$$



$$H(\mu,\mu) = 1 - \frac{1}{9}\left[1 + 2\cos\mu\right]^2$$