ECE 438 Lecture 3/4/2019 1

Two Random Variables Module 3.1.1.2

Two application scenarios

1. Input X, Output Y

2. Random signals - or random sequences

let k - time index

X_k & X_{k+l} l - integer

Joint density $f_{XY}(x,y)$ X, Y - two r.v.'s

$$P(\{a < X \leq b\} \cap \{c < Y \leq d\}) =$$

$$\int_{x=a}^{x=b} \int_{y=c}^{y=d} f_{XY}(x,y)\,dx\,dy$$

Properties of joint density

1. $f_{XY}(x,y) \geq 0$

2. $\iint_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy = 1$

joint distribution function

$$\underline{F_{XY}(x,y)} = P(\{X \le x\} \cap \{Y \le y\})$$

$$= \int_{\xi=-\infty}^{\xi=x} \int_{\eta=-\infty}^{\eta=y} \underline{f_{XY}(\xi,\eta)}\, d\xi\, d\eta$$

$$\therefore \frac{\partial^2}{\partial x \partial y}\{F_{XY}(x,y)\} = f_{XY}(x,y)$$

What is $\int_{x=-\infty}^{\infty} f_{XY}(\underset{\substack{\uparrow \\ x}}{x},y)\, dx = ?\ f_Y(y)$

similarly $\int_{y=-\infty}^{\infty} f_{XY}(x,y)\, dy = f_X(x)$

$\Uparrow$ joint density function or bivariate density function

$\Uparrow$ marginal density functions

it follows that

$F_{XY}(\infty,y) = f_Y(y)$

$F_{XY}(x,\infty) = f_X(x)$

Expectation for two random variables

Let $g(x,y)$ be a function of two real variables x & y, then

$$E\{g(\underline{X},\underline{Y})\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\, f_{XY}(x,y)\,dx\,dy$$

linearity

$$E\{\underline{a}g(X,Y) + b\,h(X,Y)\} = a\,E\{g(x,y)\} + b\,E\{h(X,Y)\}$$

Special cases

Correlation $g(x,y) = xy$

$$\overline{XY} = \underset{\uparrow \text{ expectation}}{\underline{\varepsilon}}\{XY\} = \iint xy\, f_{XY}(x,y)\,dx\,dy$$

Covariance

$$g(x,y) = (x-\overline{X})(y-\overline{Y})$$

$$= \iint \underline{(x-\overline{X})(y-\overline{Y})} f_{XY}(x,y)\,dx\,dy$$

define $\sigma_{XY}^2 = E((X-\overline{X})(Y-\overline{Y}))$

consider

$E\{(X-\overline{X})(Y-\overline{Y})\} =$

$E\{XY\} - E\{X\}\overline{Y} - \overline{X}E\{Y\} + (\overline{X}\,\overline{Y})$

$= \overline{XY} - \overline{X}\,\overline{Y} - \overline{X}\,\overline{Y} + \overline{X}\,\overline{Y}$

$= \overline{XY} - \overline{X}\,\overline{Y}$

$\therefore \sigma^2_{XY} = \overline{XY} - \overline{X}\,\overline{Y}$

<u>Correlation coefficient</u>

$$\rho_{XY} = \frac{\sigma^2_{XY}}{\sigma_X \sigma_Y}$$

can show that $0 \le |\rho_{XY}| \le 1$

<u>Independence of two random variables</u>

X & Y are statistically independent $\Longleftrightarrow$ iff

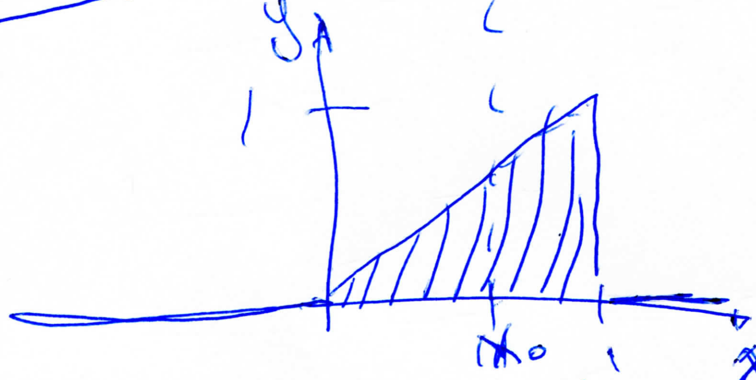
$$f_{XY}(x,y) = f_X(x)\, f_Y(y)$$

$\Rightarrow \overline{XY} = \overline{X}\,\overline{Y} \Longrightarrow \sigma^2_{XY} = 0 \Rightarrow \rho_{XY} = 0$

Converse is not true

i.e. $\rho_{XY} = 0 \not\Rightarrow X \;\&\; Y$ are independent

Example



$$f_{XY}(x,y) = \begin{cases} 2, & 0 \le y \le x \le 1 \\ 0, & \text{else} \end{cases}$$

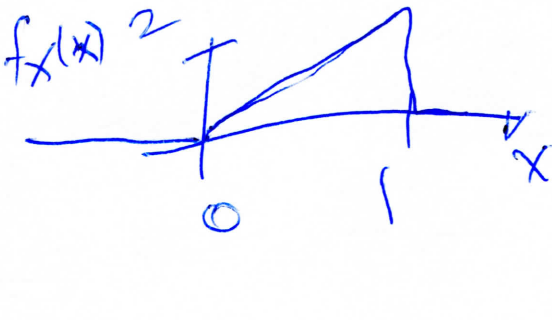
By inspection $f_{XY}(x,y) \ge 0$

$$\iint f_{XY}(x,y)\,dx\,dy = 1$$

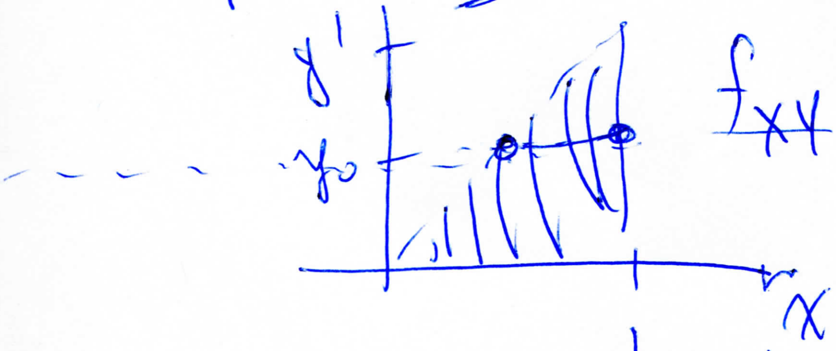
so $f_{XY}(x,y)$ is a valid density function

marginal density functions

$$f_X(x) = \int_{y=-\infty}^{\infty} f_{XY}(x,y)\,dy = \begin{cases} 2x, & 0 \le x \le 1 \end{cases}$$



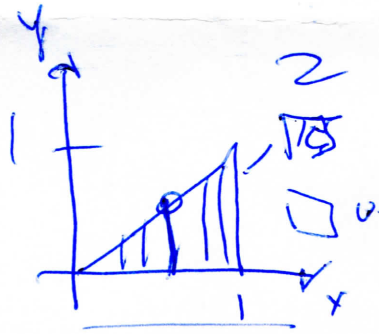
$$f_Y(y) = \int_{x=-\infty}^{\infty} f_{XY}(x,y)\,dx$$



$$f_{XY}(x,y) = \begin{cases} 2 \\ 0 \end{cases}$$

$$f_Y(y) = \begin{cases} 2(1-y), & 0 \le y \le 1 \\ 0, & \text{else} \end{cases}$$

$$\overline{XY} = \iint xy\, f_{XY}(x,y)\,dx\,dy$$

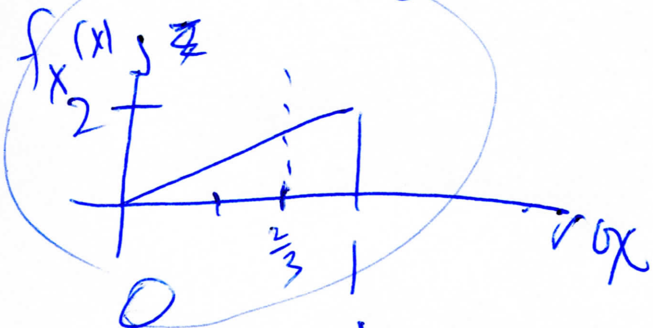


$$= \int_{x=0}^{1} \int_{y=0}^{x} xy\; 2\; dx\,dy$$

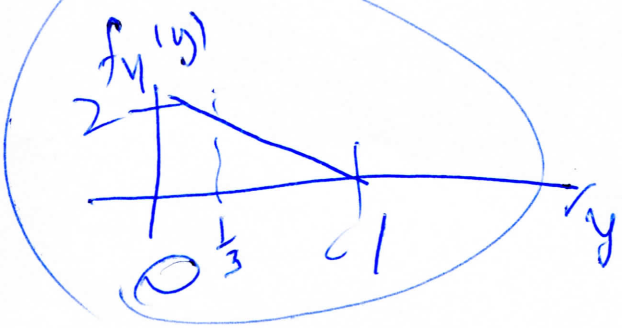
$$= \int_{x=0}^{1} x \left\{ \int_{y=0}^{x} 2y\, dy \right\} dx$$

$$= \int_{x=0}^{1} x \left(y^2 \Big|_0^x \right) dx = \int_{x=0}^{1} x^3\, dx$$

$$\overline{XY} = \Big|_0^1 \frac{x^4}{4} = \frac{1}{4}$$



$$\overline{X} = \int_0^1 x f_X(x)\,dx = \int_0^1 2x^2\,dx = 2\frac{x^3}{3}\Big|_0^1 = \frac{2}{3}$$



$$\overline{Y} = \cancel{2}\ \frac{1}{3}$$

$$\overline{X^2} = \int x^2 f_X(x)\,dx = \int_0^1 2x^3\,dx = 2\frac{x^4}{4}\Big|_0^1 = \frac{1}{2}$$

$$\therefore\ \sigma_X^2 = \frac{1}{2} - \left(\frac{2}{3}\right)^2 = \cancel{4}$$

$$= \frac{1}{2} - \frac{4}{9} = \frac{9-8}{18} = \frac{1}{18}$$

$$\sigma_Y^2 = \frac{1}{18}$$