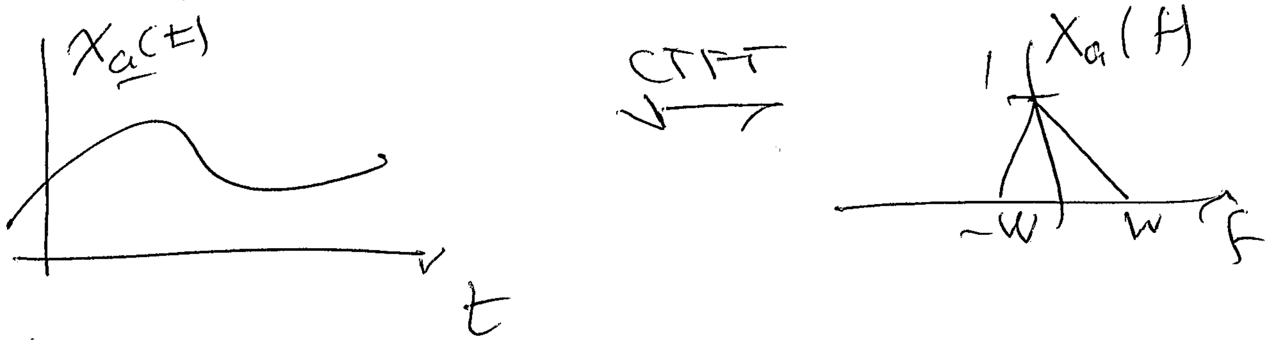
ECE 438 Lecture 2/4/2019

Relation between CTFT & DTFT
(Module 1.4.2)

Time Domain Frequency Domain



know that $X_s(f) = \frac{1}{T} \sum_k X_a(f - \frac{k}{T})$

have $x_s(t) = \sum_n x_a(nT)\delta(t - nT)$

by direct CTFT

$$X_s(f) = \int_{-\infty}^{\infty} x_s(t) e^{-j2\pi ft} dt$$

$$X_s(f) = \sum_n x_a(nT)\int_{-\infty}^{\infty}\delta(t-nT)\,e^{-j2\pi ft}\,dt$$ ②

$$\boxed{X_s(f) = \sum_n x_a(nT)\,e^{-j2\pi f nT}}$$

$$\boxed{X_d(\omega) = \sum_n x_d[n]\,e^{-j\omega n}}$$

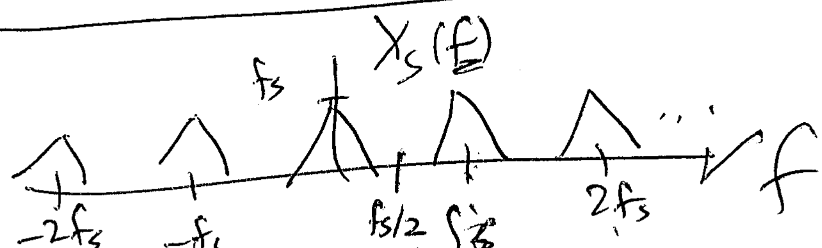
$$x_d[n] = x_a(nT)$$

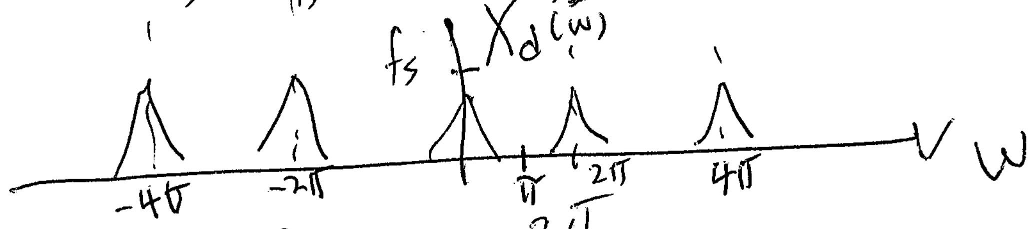
$$X_s(f) = \sum_n x_d[n]\,e^{-j2\pi f nT}$$

let $\boxed{2\pi f T = \omega}$ $\quad f = \frac{\omega}{2\pi}\frac{1}{T} = \frac{\omega}{2\pi} f_s$

$$\boxed{X_d(\omega) = \cancel{\sum_n x_d[n]} = X_s\left(\frac{\omega}{2\pi} f_s\right)}$$

$$\omega = 2\pi\left(\frac{f}{f_s}\right)$$





when $f = f_s$ $\quad \omega = 2\pi$

Example:

$$x_a(t) = \cos(2\pi(1000)t)$$

$$X_a(f) = \frac{1}{2}\left[\delta(f-1000) + \delta(f+1000)\right]$$

let $f_s = 4\,\text{kHz}$

$$\cancel{X_d(\omega)} = \qquad X_s(f) = 4000 \sum_k X_a(f - 4000k)$$

$$X_s(f) = 4000 \sum_k \frac{1}{2}\left[\delta(f - 1000 - 4000k) + \delta(f + 1000 - 4000k)\right]$$

$$\boxed{X_d(\omega) = 2000 \sum_k \delta\left(\frac{\omega}{2\pi}4000 - 1000 - 4000k\right) + \delta\left(\frac{\omega}{2\pi}4000 + 1000 - 4000k\right)}$$

know $X_d(\omega) = \pi\{\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\} \quad |\omega| < \pi$

note $X_d(\omega + 2\pi l) = X_d(\omega) \quad l$ - integer

$$\omega_0 = 2\pi \frac{\text{rad.}}{\text{cycle}} \cdot 1000 \frac{\text{cycles}}{\text{sec}} \cdot \frac{1}{f_s} \frac{\text{sec.}}{\text{sample}} \qquad \frac{1}{4000}$$

$$\omega = 2\pi \frac{1000}{4000} \frac{\text{rad}}{\text{sample}}$$

$$= \frac{\pi}{2} \frac{\text{rad.}}{\text{sample}}$$

$$\boxed{X_d(\omega) = \pi\left\{\delta\left(\omega - \frac{\pi}{2}\right) + \delta\left(\omega + \frac{\pi}{2}\right)\right\} \quad |\omega| < \pi}$$

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relation

CT only $\boxed{\delta(ax+b) = \frac{1}{|a|}\,\delta\left(x + b/a\right)}$ $\qquad a = \frac{4000}{2\pi}$

$$X_d(\omega) = \frac{2000 \cdot 2\pi}{4000} \sum_k \left\{ \delta\left(\omega + \frac{-1000 - 4000k}{4000/2\pi}\right) + \delta\left(\omega - \frac{+1000 - 4000k}{4000/2\pi}\right) \right\}$$

$$= \pi \sum_k \left[\delta\left(\omega - \frac{\pi}{2} - 2\pi k\right) + \delta\left(\omega + \frac{\pi}{2} - 2\pi k\right) \right]$$

Scaling in DT (Sampling Rate Conversion)

Module 1.4.3

DT scaling

down sampling: $x[n] \rightarrow \boxed{D\downarrow} \rightarrow y[n] = x[Dn]$ $\qquad D$ - integer

upsampling: $x[n] \rightarrow \boxed{D\uparrow} \rightarrow y[n] = \begin{cases} x[n/D], & n/D \text{ - integer} \\ 0, & \text{else} \end{cases}$

Recall for CTFT

$$x(at) \xleftrightarrow{CTFT} \frac{1}{|a|} X(f/a)$$

downsampling

$$y[n] = x[Dn]$$

$$Y(\omega) = \sum_n x[Dn] e^{-j\omega n}$$

want to express $Y(\omega)$ in terms of $X(\omega)$

$$X(\omega) = \sum_m x[m] e^{-j\omega m}$$

let $m = Dn \Rightarrow n = \frac{m}{D}$

$$\cancel{Y(\omega) = \sum_{m=-\infty}^{\infty} x[m] e^{-j\omega\left(\frac{m}{D}\right)}}$$

let $S_D[m] = \begin{cases} 1, & m/D - \text{integer} \\ 0, & \text{else} \end{cases}$

$$Y(\omega) = \sum_m S_D[m]\, x[m]\, e^{-j\omega\left(\frac{m}{D}\right)}$$

Trick $\boxed{S_D[m] = \frac{1}{D}\sum_{k=0}^{D-1} e^{-j2\pi k m/D}}$

$$= \frac{1}{D}\left[\frac{1 - e^{-j2\pi m}}{1 - e^{-j2\pi m/D}}\right].$$

have $S_D[m] = \begin{cases} 1, & m/D - \text{integer} \\ 0, & \text{else} \end{cases}$

$$Y(\omega) = \frac{1}{D}\sum_{k=0}^{D-1}\sum_m x[m]\left[e^{-j\omega\frac{m}{D}} e^{-j2\pi k m/D}\right]$$

$$= \frac{1}{D}\sum_{k=0}^{D-1}\sum_m x[m]\, e^{-j\left(\frac{\omega - 2\pi k}{D}\right)m}$$

$$Y(\omega) = \frac{1}{D}\sum_{k=0}^{D-1} X\left(\frac{\omega - 2\pi k}{D}\right)$$

recall $y[n] = x[Dn]$ combines sampling & scaling

<u>Upsampling</u>

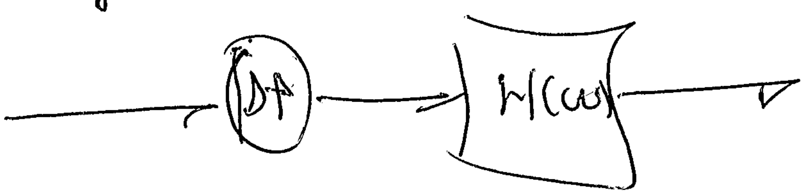
$$y[n] = \begin{cases} x[n/D], & n/D \text{ - integer} \\ 0, & \text{else} \end{cases}$$

$$Y(\omega) = X(D\omega)$$

Decimator

lower sampling rate

Interpolator



raise sampling rate

Multirate filtering