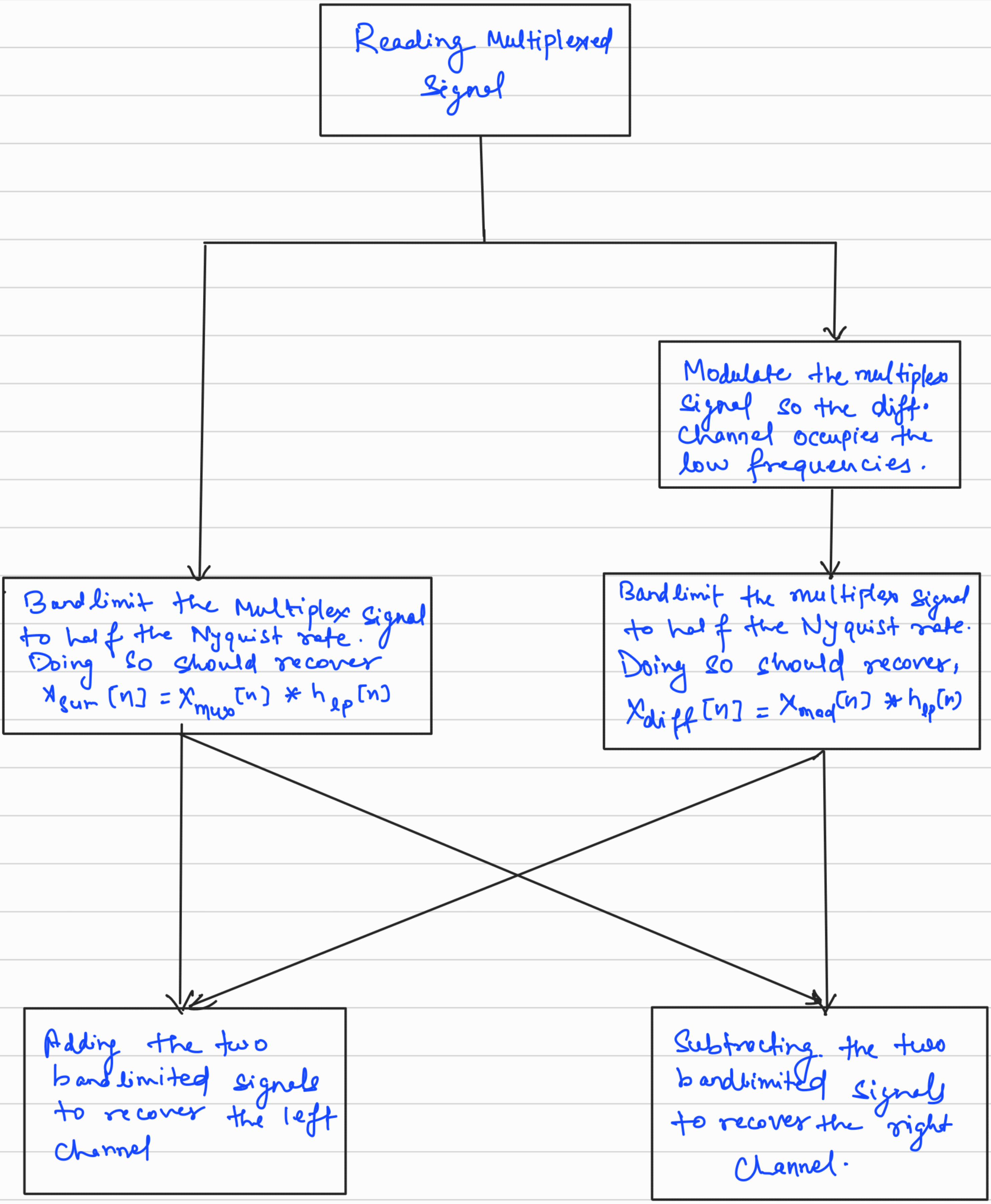
Prob. 1

Part (b)

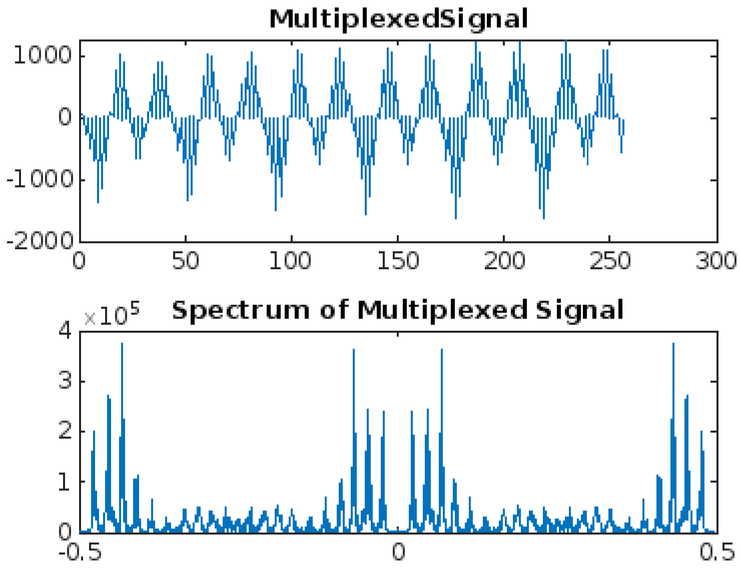

Reading Multiplexed Signal
Modulate the multiplex signal so the diff. Channel occupies the low frequencies.
Bandlimit the Multiplex signal to half the Nyquist rate. Doing so should recover $x_{sum}[n] = x_{mux}[n] * h_{lp}[n]$
Bandlimit the multiplex signal to half the Nyquist rate. Doing so should recover, $x_{diff}[n] = x_{mod}[n] * h_{lp}[n]$
Adding the two bandlimited signals to recover the left Channel
Subtracting the two bandlimited signals to recover the right Channel.

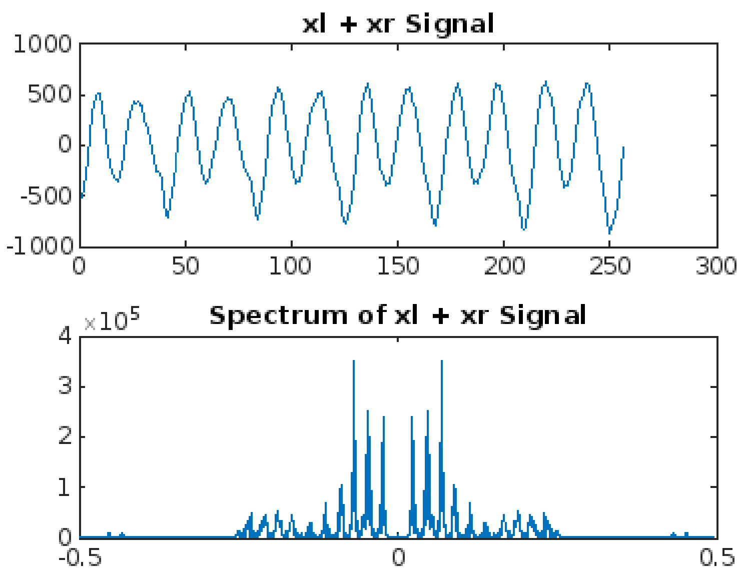
Contents

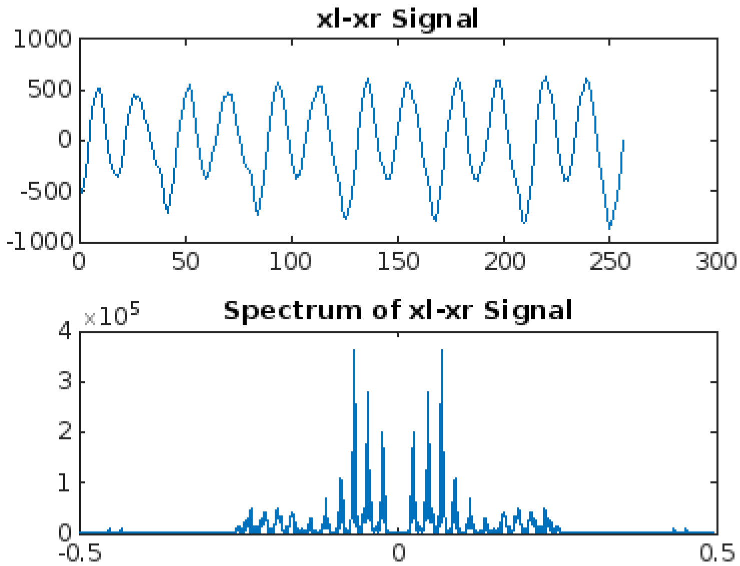
Verifying the results match with the original multiplexed signal

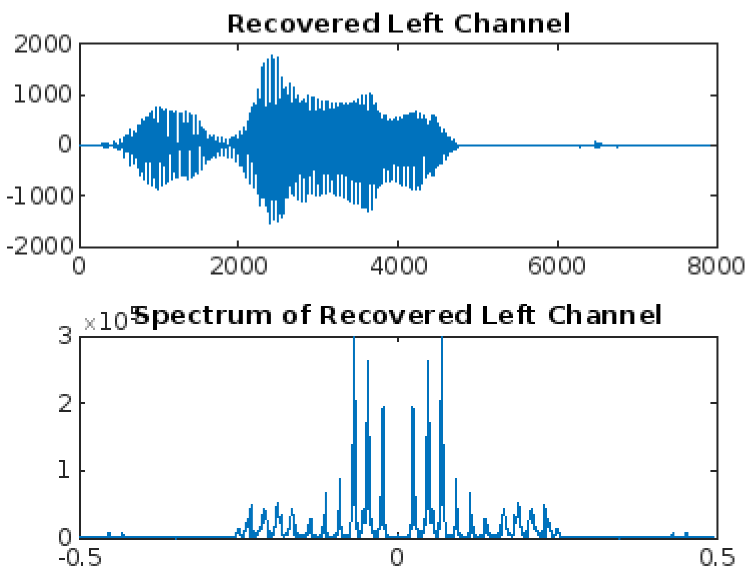
```
x_mux = xmux;
% Obtaining the length of the signal
N = length(x_mux);
% Computing the number of points required for the FFT (2^k) > N
N_spect = pow2(ceil(log2(N)));
% Number of points to show the plot
N_plot = 256;

del_mu = 1 / N_spect;
mu = - 0.5:del_mu:(0.5 - del_mu);
% input('Multiplexed Signal');
soundsc(x_mux);
X_mux = fft(x_mux, N_spect);
X_mux = fftshift(X_mux);
X_mux_mag = abs(X_mux);
x_mux_clip = x_mux(round(N ./ 2):round(N ./ 2 + N_plot - 1));
% Plotting the Result
figure(2);
subplot(2 , 1 , 1);
plot(x_mux_clip);
title('MultiplexedSignal');
subplot(2 , 1 , 2);
plot(mu , X_mux_mag);
title('Spectrum of Multiplexed Signal');

% Bandlimiting the input to have the Nyquist Rate
x1 = conv(x_mux , h);
% input('xl+xr signal');
soundsc(x1);
X1 = fft(x1 , N_spect);
X1 = fftshift(X1);
X1_mag = abs(X1);
x1_clip= x1(round(N ./ 2):round(N ./ 2 + N_plot-1));
% Plotting the Results
figure(3);
subplot(2 , 1 , 1);
plot(x1_clip);
title('xl + xr Signal');
subplot(2 , 1 , 2);
plot(mu, X1_mag);
title('Spectrum of xl + xr Signal');
% Shiftingthe frequency of the multiplexed signal to recover the xl-xr
% channel utiliting the LPF
omega0 = pi;
n = 0:1:(N - 1);
x_mod = x_mux .* cos(omega0 .* n);
x2 = conv(h , x_mod);
% input('xl-xr signal');
```

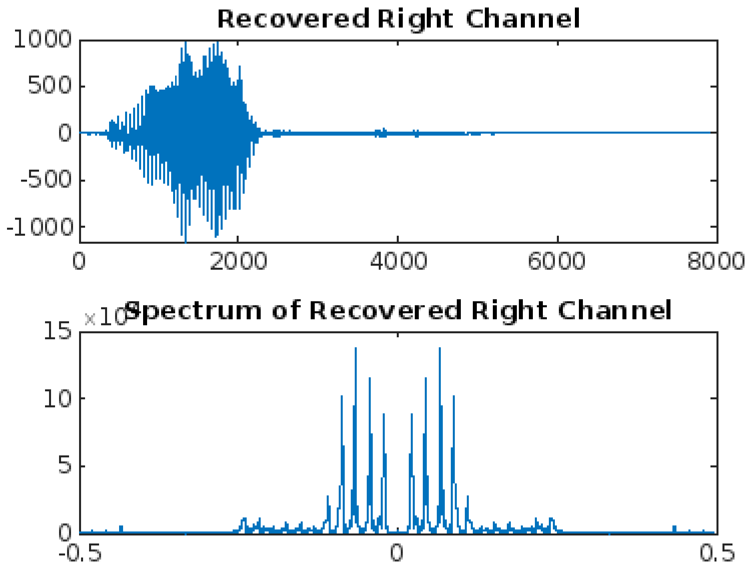
```
soundsc(x2);
X2 = fft(x2 , N_spect);
X2 = fftshift(X2);
X2_mag = abs(X2);
x2_clip = x2(round(N ./ 2):round(N ./ 2 + N_plot-1));
% Plotting the results
figure(4);
subplot(2 , 1 , 1);
plot(x2_clip);
title('xl-xr Signal');
subplot(2 , 1 , 2);
plot(mu, X2_mag);
title('Spectrum of xl-xr Signal');

%Adding the two signals together to recover xl
xl = (x1 + x2) / 2;
% input('recovered left channel');
soundsc(xl)
XL = fft(xl , N_spect);
XL = fftshift(XL);
XL_mag = abs(XL);
% Plotting the results
figure(5);
subplot(2 , 1 , 1);
plot(xl);
title('Recovered Left Channel');
subplot(2 , 1 , 2);
plot(mu, XL_mag);
title('Spectrum of Recovered Left Channel');

% Subtracting the two signals together to recover xr
xr = (x1-x2) / 2;
% input('recovered right channel');
soundsc(xr)
XR = fft(xr , N_spect);
XR = fftshift(XR);
XR_mag = abs(XR);
% Plotting the results
figure(6);
subplot(2 , 1 , 1);
plot(xr);
title('Recovered Right Channel');
subplot(2 , 1 , 2);
plot(mu, XR_mag);
title('Spectrum of Recovered Right Channel');
```


MultiplexedSignal
1000
0
-1000
-2000
0
50
100
150
200
250
300
×10^5
Spectrum of Multiplexed Signal
4
3
2
1
0
-0.5
0
0.5


xl + xr Signal
1000
500
0
-500
-1000
0
50
100
150
200
250
300
×10^5
Spectrum of xl + xr Signal
4
3
2
1
0
-0.5
0
0.5


xl-xr Signal
1000
500
0
-500
-1000
0
50
100
150
200
250
300
Spectrum of xl-xr Signal
$\times 10^5$
4
3
2
1
0
-0.5
0
0.5

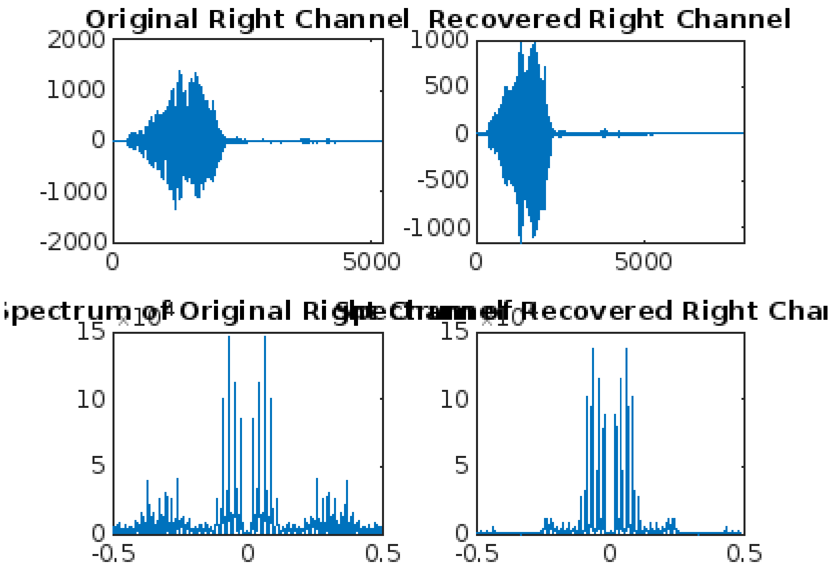

Recovered Left Channel
2000
1000
0
-1000
-2000
0
2000
4000
6000
8000
Spectrum of Recovered Left Channel
$\times 10^5$
3
2
1
0
-0.5
0
0.5

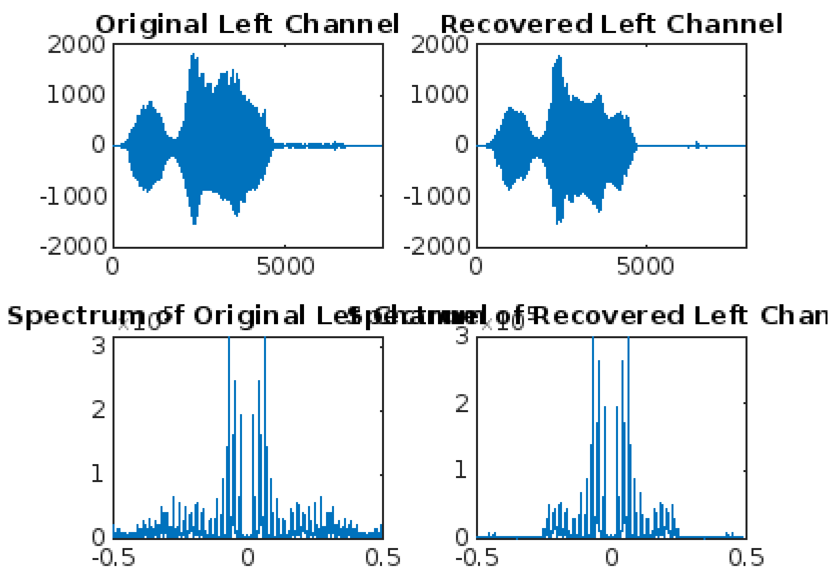


Comparison of theOriginal and Recovered Signals

```
% Right Channel Comparison
xr_Orig = load('ysf1s1t0.txt')';
figure(7)
subplot(2 , 2 , 1)
plot(xr_Orig);
title('Original Right Channel');
XR_Orig = fft(xr_Orig , N_spect);
XR_Orig = fftshift(XR_Orig);
XR_Orig_mag = abs(XR_Orig);
subplot(2 , 2 , 3)
plot(mu , XR_Orig_mag);
title('Spectrum of Original Right Channel');
subplot(2 , 2 , 2)
plot(xr);
title('Recovered Right Channel');
subplot(2 , 2 , 4)
plot(mu , XR_mag) ;
title('Spectrum of Recovered Right Channel');
% Left Channel Comparison
xl_Orig = load('erf1s1t0.txt')';
figure(8)
subplot(2 , 2 , 1)
plot(xl_Orig);
title('Original Left Channel');
XL_Orig = fft(xl_Orig , N_spect);
XL_Orig = fftshift(XL_Orig);
XL_Orig_mag = abs(XL_Orig);
subplot(2 , 2 , 3)
plot(mu ,XL_Orig_mag);
title('Spectrum of Original Let Channel');
subplot(2 , 2 , 2)
plot(xl);
title('Recovered Left Channel');
subplot(2 , 2 , 4)
```

```
plot(mu , XL_mag) ;
title('Spectrum of Recovered Left Channel');
```



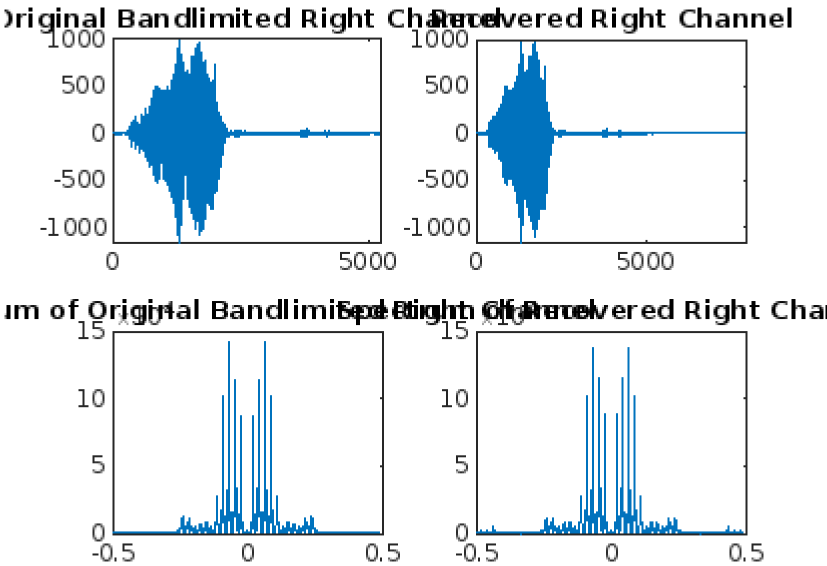


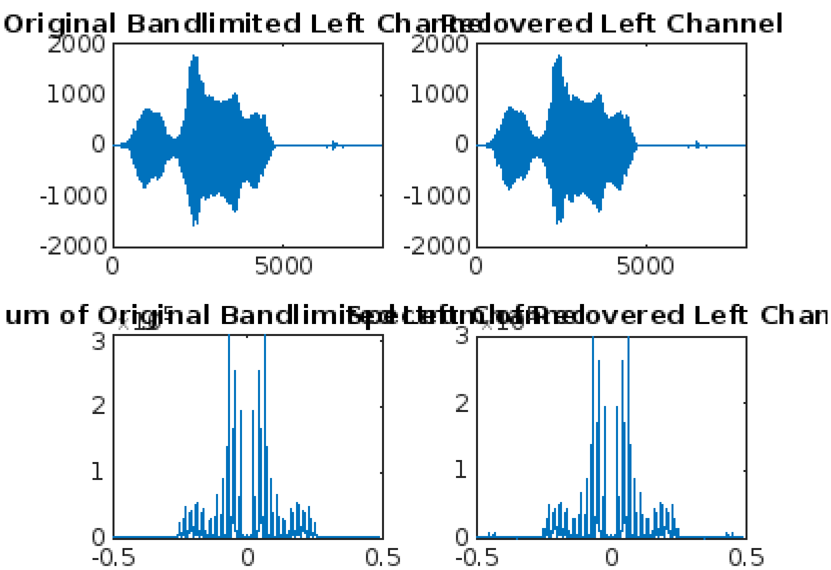
Comparison of Original Bandlimited and Recovered Signals

Right Channel Comparison

```
xr_Orig = conv(h , xr_Orig);
figure(9);
subplot(2 , 2 , 1);
plot(xr_Orig);
title('Original Bandlimited Right Channel');
XR_Orig = fft(xr_Orig , N_spect);
XR_Orig = fftshift(XR_Orig);
XR_Orig_mag = abs(XR_Orig);
subplot(2 , 2, 3);
plot(mu , XR_Orig_mag);
```

```
title('Spectrum of Original Bandlimited Right Channel');
subplot(2 , 2 , 2);
plot(xr);
title('Recovered Right Channel');
subplot(2 , 2 , 4);
plot(mu , XR_mag);
title('Spectrum of Recovered Right Channel');
% LeftChannel Comparison
xl_Orig = conv(h , xl_Orig);
figure(10);
subplot(2 , 2 , 1);
plot(xl_Orig);
title('Original Bandlimited Left Channel');
XL_Orig = fft(xl_Orig , N_spect);
XL_Orig = fftshift(XL_Orig);
XL_Orig_mag = abs(XL_Orig);
subplot(2 , 2, 3);
plot(mu, XL_Orig_mag);
title('Spectrum of Original Bandlimited Left Channel');
subplot(2 , 2 , 2);
plot(xl);
title('Recovered Left Channel');
subplot(2 , 2 , 4);
plot(mu , XL_mag);
title('Spectrum of Recovered Left Channel');
```





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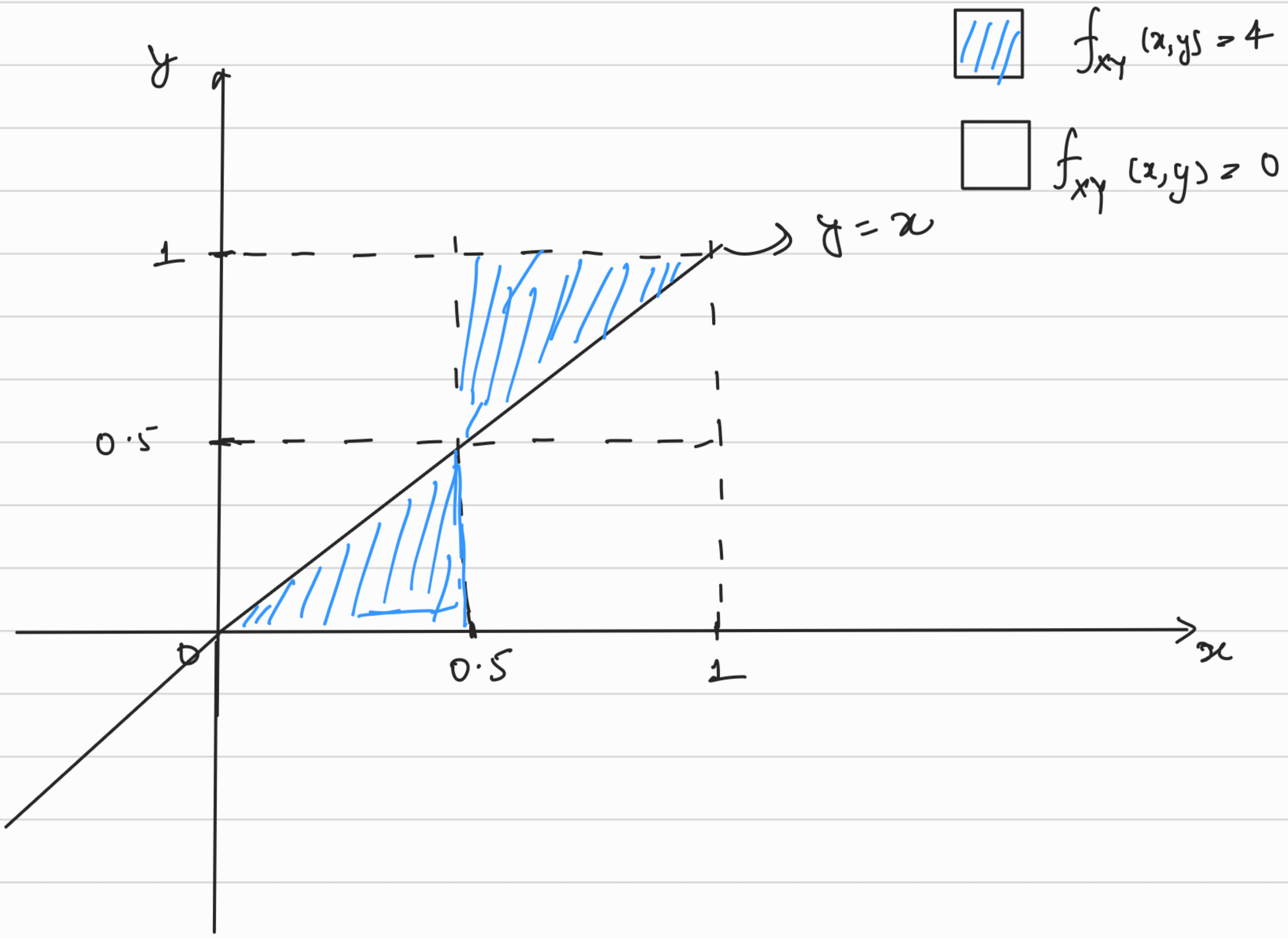
(e) When playing back the reconstructed signals, the sound is similar to the original bandlimited signals. However, there is a high pitched noise present in the reconstructed signal as it can be seen in the plots of the recovered signals. The high-frequency components are still present in the spectrum of the recovered signals.

Prob. 2 Given:

$$f_{XY}(x,y) = \begin{cases} 4, & 0 \le x \le 0.5, \quad y \le x \\ 4, & 0.5 \le x \le 1, \quad x \le y \le 1 \\ 0, & \text{else.} \end{cases}$$



(b) $f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\, dy$

for $0 \le x \le 0.5$: $f_X(x) = \int_0^x 4\, dy = 4x$

for $0.5 \leq x \leq 1$: $\quad f_X(x) = \int_x^1 4\,dy = 4(1-x)$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx$$

for $0 \leq y \leq 0.5$: $f_Y(y) = \int_y^{0.5} 4\,dx = 4(0.5-y)$

for $0.5 \leq y \leq 1$: $f_Y(y) = \int_{0.5}^{y} 4\,dx = 4(y-0.5)$.

(c) No $f_X(x)$ & $f_Y(y)$ are not independent; since $f_{XY}(x,y) \neq f_X(x)\, f_Y(y)$

d) $$E[X] = \int_{-\infty}^{\infty} x\, f_X(x)\,dx$$

$$= \int_0^{0.5} x\,4x\,dx + \int_{0.5}^{1} x\,4(1-x)\,dx$$

$$= \left|4\frac{x^3}{3}\right|_0^{0.5} + \left|\frac{4x^2}{2} - \frac{4x^3}{3}\right|_{0.5}^{1}$$

$$= 4\times\frac{1}{24} + \frac{4}{2} - \frac{4}{3} - 4\times\frac{1}{8} + 4\times\frac{1}{24}$$

$$= \frac{1}{6} + 2 - \frac{4}{3} - \frac{1}{2} + \frac{1}{6}$$

$$= \frac{1}{3} + \frac{2}{3} - \frac{1}{2}$$

$$= 1 - \frac{1}{2}$$

$$= \frac{1}{2}$$

$$E[Y] = \int_{-\infty}^{\infty} y\, f(y)\, dy$$

$$= \int_{0}^{0.5} y(2-4y)\, dy + \int_{0.5}^{1} 4y(y-0.5)\, dy$$

$$= \left| \frac{2y^2}{2} - \frac{4y^3}{3} \right|_0^{0.5} + \left| \frac{4y^3}{3} - \frac{2y^2}{2} \right|_{0.5}^{1}$$

$$= 2\times\frac{1}{4}\times\frac{1}{2} - 4\times\frac{1}{8}\times\frac{1}{3} + \frac{4}{3} - \frac{2}{2} - 4\times\frac{1}{8}\times\frac{1}{3} + 2\times\frac{1}{8}$$

$$= \frac{1}{4} - \frac{1}{6} + \frac{4}{3} - 1 - \frac{1}{6} + \frac{1}{4}$$

$$= \frac{1}{2} - \frac{1}{3} + \frac{1}{3}$$

$$= \frac{1}{2}.$$

<u>For</u> <u>variance</u>

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx$$

$$= \int_{0}^{0.5} x^2 4x\, dx + \int_{0.5}^{1} x^2\, 4(1-x)\, dx$$

$$= \int_{0}^{0.5} 4x^3\, dx + \int_{0.5}^{1} (4x^2 - 4x^3)\, dx$$

$$= \left| 4\frac{x^4}{4} \right|_0^{0.5} + \left| \frac{4x^3}{3} - \frac{4x^4}{4} \right|_{0.5}^{1}$$

$$= \frac{1}{16} + \frac{4}{3} - \frac{4}{4} - 4 \times \frac{1}{24} + \frac{4}{64}$$

$$= \frac{1}{16} + \frac{4}{3} - 1 - \frac{1}{6} + \frac{1}{16}$$

$$= \frac{1}{8} + \frac{1}{3} - \frac{1}{6}$$

$$= \frac{1}{8} + \frac{1}{6}$$

$$= \frac{6+8}{48}$$

$$= \frac{14}{48} = \frac{7}{24}$$

$$E(Y^2) = \int_{-\infty}^{\infty} y^2 f_Y(y)\, dy$$

$$= \int_0^{0.5} y^2\, 4(0.5-y)\, dy + \int_{0.5}^{1} 4y^2 (y-0.5)\, dy$$

$$= \int_0^{0.5} (2y^2 - 4y^3)\, dy + \int_{0.5}^{1} (4y^3 - 2y^2)\, dy$$

$$= \left| \frac{2y^3}{3} - \frac{4y^4}{4} \right|_0^{0.5} + \left| \frac{4y^4}{4} - \frac{2y^3}{3} \right|_{0.5}^{1}$$

$$= 2 \times \frac{1}{24} - \frac{4}{64} + \frac{4}{4} - \frac{2}{3} - \frac{4}{64} + \frac{2}{24}$$

$$= \frac{1}{12} - \frac{1}{16} + 1 - \frac{2}{3} - \frac{1}{16} + \frac{1}{12}$$

$$= \frac{1}{6} - \frac{1}{8} + \frac{1}{3}$$

$$= \frac{4-3+8}{24}$$

$$= \frac{9}{24}$$

$$= \frac{3}{8}$$

So, $\sigma_x^2 = E[X^2] - (E[X])^2$

$$= \frac{7}{24} - \frac{1}{4}$$

$$= \frac{7-6}{24} = \frac{1}{24}$$

$\sigma_Y^2 = E[Y^2] - (E[Y])^2$

$$= \frac{3}{8} - \frac{1}{4}$$

$$= \frac{3-2}{8}$$

$$= \frac{1}{8}$$

$$E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_{XY}(x,y)\, dx\, dy$$

$$= \int_0^{0.5}\int_0^{x} xy\,4\,dy\,dx + \int_{0.5}^{1}\int_{x}^{1} xy\,4\,dy\,dx$$

$$= \int_0^{0.5} x \left|2y^2\right|_0^x dx + \int_{0.5}^{1} x\left(2y^2\right|_x^1 dx$$

$$= \int_0^{0.5} 2x^3\,dx + \int_{0.5}^{1} x(2-2x^2)\,dx$$

$$= \left|\frac{2\times x^4}{4}\right|_0^{0.5} + \left|\frac{2x^2}{2} - \frac{2x^4}{4}\right|_{0.5}^{1}$$

$$= 2\times\frac{1}{64} + \frac{2}{2} - \frac{2}{4} - \frac{2\times 1}{8} + 2\times\frac{1}{64}$$

$$= \frac{1}{32} + 1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{32}$$

$$= \frac{1}{16} + \frac{1}{2} - \frac{1}{4}$$

$$= \frac{1}{16} + \frac{1}{4} = \frac{5}{16}$$

So, $$\rho_{XY} = \frac{E[XY] - \mu_X\mu_Y}{\sigma_X\sigma_Y}$$

$$= \frac{\frac{5}{16} - \frac{1}{4}}{\sqrt{\frac{1}{24}} \times \sqrt{\frac{1}{8}}}$$

$$= \frac{1}{16} \times \sqrt{24 \times 8}$$

$$= \frac{1}{16} \times 8\sqrt{3}$$

$$= \frac{\sqrt{3}}{2}$$

Prob. 3.

(a) $$\sigma_x^2 = E[x^2] - \mu_x^2$$

$$E[x^2] = \sigma_x^2 + \mu_x^2$$

Similarly,

$$E[Y^2] = \sigma_Y^2 + \mu_Y^2$$

$$\rho_{xy} = \frac{E[XY] - \mu_x \mu_y}{\sigma_x \sigma_y}$$

$$\Rightarrow E[XY] = \rho_{xy}\sigma_x\sigma_y + \mu_x\mu_y$$

Now MSE.

$$\varepsilon = E[\|\hat{x} - x\|^2] = E[\|aY + b - x\|^2]$$

$$= E[a^2Y^2 + b^2 + x^2 + 2abY - 2bx - 2axY]$$

$$= a^2 E[Y^2] + b^2 + E[X^2] + 2ab\, E[Y]$$
$$- 2b\, E[X] - 2a\, E[XY]$$

$$\mathcal{E} = a^2(\sigma_Y^2 + \mu_Y^2) + b^2 + (\sigma_X^2 + \mu_X^2) + 2ab\,\mu_Y - 2b\,\mu_X$$
$$- 2a(\rho_{XY}\sigma_X\sigma_Y + \mu_X\mu_Y)$$

For minimize $\mathcal{E}$ w.r.t a & b take the derivative w.r.t a & b and then set it to 0.
(Hint: Try differentiating again & check if the function is actually convex or not).

So, $$\frac{\partial \mathcal{E}}{\partial a} = 2a(\sigma_Y^2 + \mu_Y^2) + 2b\,\mu_Y - 2(\rho_{XY}\sigma_X\sigma_Y + \mu_X\mu_Y) = 0$$

$$\Rightarrow a(\sigma_Y^2 + \mu_Y^2) + b\,\mu_Y - (\rho_{XY}\sigma_X\sigma_Y + \mu_X\mu_Y) = 0$$

$$\Rightarrow a = \frac{\rho_{XY}\sigma_X\sigma_Y + \mu_X\mu_Y - b\,\mu_Y}{\sigma_Y^2 + \mu_Y^2} \quad — (1)$$

Similarly for b,

$$\frac{\partial \mathcal{E}}{\partial b} = 2b + 2a\,\mu_Y - 2\mu_X = 0$$

$$\Rightarrow b = \mu_X - a\,\mu_Y \quad — (2)$$

Putting ② in ① we get

$$a = \frac{\rho_{xy}\,\sigma_x}{\sigma_y} \qquad -③$$

Put ③ in ② we get,

$$b = \frac{\sigma_y \mu_x - \rho_{xy}\,\sigma_x \mu_y}{\sigma_y}.$$

(b)

$$\mathcal{E} = a^2(\sigma_y^2 + \mu_y^2) + b^2 + (\sigma_x^2 + \mu_x^2) + 2ab\,\mu_y - 2b\,\mu_x$$

$$- 2a(\rho_{xy}\sigma_x\sigma_y + \mu_x\mu_y)$$

$$= \frac{\rho_{xy}^2\,\sigma_x^2}{\sigma_y^2}(\sigma_y^2 + \mu_y^2) + \left(\frac{\sigma_y \mu_x - \rho_{xy}\sigma_x\mu_y}{\sigma_y}\right)^2$$

$$+ (\sigma_x^2 + \mu_x^2) + 2\frac{\rho_{xy}\sigma_x}{\sigma_y}\left(\frac{\sigma_y\mu_x - \rho_{xy}\sigma_x\mu_y}{\sigma_y}\right)\mu_y$$

$$- 2\left(\frac{\sigma_y\mu_x - \rho_{xy}\sigma_x\mu_y}{\sigma_y}\right)\mu_x - 2\frac{\rho_{xy}\sigma_x}{\sigma_y}\left(\rho_{xy}\sigma_x\sigma_y + \mu_x\mu_y\right)$$

$$= \sigma_x^2 - \rho_{xy}^2\,\sigma_x^2$$

(c) (i) $\rho_{XY} = 0$.

$a = 0$, $b = \mu_X$ & $\mathcal{E} = \sigma_X^2$

This tells us one thing for sure that our estimate would be equal to the mean of the X.

i.e.

$$\hat{X} = \mu_X .$$

(ii) $\rho_{XY} = 1$

$$a = \frac{\sigma_X}{\sigma_Y} \qquad b = \frac{\sigma_Y \mu_X - \sigma_X \mu_Y}{\sigma_Y} = \mu_X - a\mu_Y$$

and $\mathcal{E} = 0$, the error goes to zero.

(iii) $\rho_{XY} = -1$

$$a = -\frac{\sigma_X}{\sigma_Y} \qquad b = \frac{\sigma_Y \mu_X + \sigma_X \mu_X}{\sigma_Y} = \mu_X - a\mu_Y$$

and $\mathcal{E} = 2\sigma_X^2$

The error becomes twice of σ_X^2.

Prob. 4 :

Given

$$\gamma_{xx}[n] = \begin{cases} 1, & n=0 \\ \frac{3}{4}, & |n|=1 \\ \frac{1}{4}, & |n|=2 \\ 0, & \text{else.} \end{cases}$$

$x[n] \rightarrow \mu_x = 0$

$$y[n] = \frac{1}{2}x[n] + \frac{1}{4}x[n-1].$$

(a) $E[y[n]] = E\{\frac{1}{2}x[n] + \frac{1}{4}x[n-1]]$

$= \frac{1}{2}E[x[n]] + \frac{1}{4}E[x[n-1])$

$= \frac{1}{2}\mu_x + \frac{1}{4}\mu_x \rightarrow (\because \text{WSS})$

$= 0$

(b) $\gamma_{xy}[n] = E[x[m]y[m+n]]$

$= E[x[m](\frac{1}{2}x[m+n] + \frac{1}{4}x[m+n-1])]$

$= E[\frac{1}{2}x[m]x[m+n] + \frac{1}{4}x[m]x[m+n-1])$

$$= \frac{1}{2} r_{xx}[n] + \frac{1}{4} r_{xx}[n-1].$$

$$= \left(\frac{1}{2}\delta[n] + \frac{1}{4}\delta[n-1]\right) * r_{xx}[n].$$

n	≤ -3	-2	-1	0	1	2	3	≥ 4
$\frac{1}{2} r_{xx}[n]$.	0	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$	0	0
$\frac{1}{4} r_{xx}[n-1]$	0	0	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{12}$	0
$r_{xy}[n]$	0	$\frac{1}{6}$	$\frac{5}{12}$	$\frac{2}{3}$	$\frac{7}{12}$	$\frac{1}{3}$	$\frac{1}{12}$	0

(c)

$$h[n] = \frac{1}{2}\delta[n] + \frac{1}{4}\delta[n-1].$$

$$r_{yy}[n] = h[-n] * r_{xy}[n].$$

$$= \frac{1}{2} r_{xy}[n] + \frac{1}{4} r_{xy}[n+1].$$

n	≤ -4	-3	-2	-1	0	1	2	3	≥ 4
$\frac{1}{2} r_{xy}[n]$	0	0	$\frac{1}{12}$	$\frac{5}{24}$	$\frac{1}{3}$	$\frac{7}{24}$	$\frac{1}{6}$	$\frac{1}{24}$	0

$\frac{1}{4} r_{xy}[n+1]$	0	$\frac{1}{24}$	$\frac{5}{48}$	$\frac{1}{6}$	$\frac{7}{48}$	$\frac{1}{12}$	$\frac{1}{48}$	0	0
$r_{yy}[n]$	0	$\frac{1}{24}$	$\frac{3}{16}$	$\frac{3}{8}$	$\frac{23}{48}$	$\frac{3}{8}$	$\frac{3}{16}$	$\frac{1}{24}$	0