1 May 202 **Name:_______________**

ECE 438 **Final Exam** **Spring 2023**

- You have 120 minutes to work the following **five** problems.
- Be sure to show all your work to obtain full credit.
- The exam is closed book and closed notes.
- Calculators, smart phones, and smart watches are not permitted, and must be put away.

1. (25 pts.) Consider the linear, time-invariant system defined by the difference equation

$$y[n]=\frac{1}{2}\left\{x[n]+x[n-2]\right\}.$$

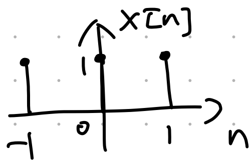
a. (5) Find the response to the input $x[n]=\begin{cases} 1, & -1\le n\le 1 \\ 0, & \text{else}\end{cases}$

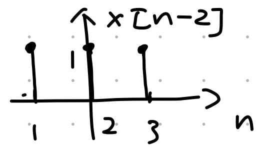
b. (10) Find a simple expression for the frequency response $H(\omega)$ of this system.

c. (4) Find a simple expression for the magnitude $|H(\omega)|$ of the frequency response.

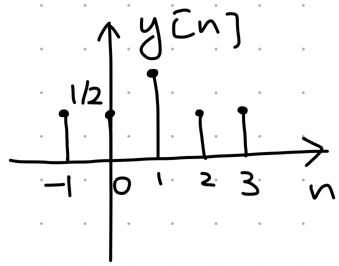
d. (3) Find a simple expression for the phase $\angle H(\omega)$ of the frequency response.

e. (3) Carefully sketch $|H(\omega)|$ and $\angle H(\omega)$. Be sure to dimension all important quantities on both the horizontal and vertical axes.





a) $y[n] = \frac{1}{2}\{x[n] + x[n-2]\}$



$$= \begin{cases} \frac{1}{2} & -1 \le n \le 0 \text{ or } 2 \le n \le 3 \\ 1 & n = 1 \\ 0 & \text{else} \end{cases}$$

b) Method - 1

let $x[n] = e^{j\omega n}$

$$y[n] = H(\omega) \cdot x[n] = \frac{1}{2}\{x[n] + x[n-2]\}$$

$$= \frac{1}{2}\left(e^{j\omega n} + e^{j\omega(n-2)}\right)$$

$$= \frac{1}{2} e^{j\omega n}\left(1 + e^{-j2\omega}\right)$$

$$H(\omega) = \frac{1}{2}\left(1 + e^{-j2\omega}\right) = \frac{1}{2} e^{-j\omega}\left(e^{j\omega} + e^{-j\omega}\right)$$

$$= e^{-j\omega}\cos(\omega)$$

Method - 2

$$y[n] = h[n] * x[n]$$

$$\Rightarrow \quad h[n] = \frac{1}{2}\{\delta[n] + \delta[n-2]\}$$

$\updownarrow$ DTFT

$$H(\omega) = \frac{1}{2}\{1 + e^{-j2\omega}\} = e^{-j\omega}\cos(\omega)$$

5 pt : correct expression of $H(\omega)$ before simplification

3pt : using Euler's formula

2pt : correct expression of $H(\omega)$ after simplification

Method - 3 z-transform.

$$Y_{ZT}(z) = \frac{1}{2}\{X_{ZT}(z) + X_{ZT}(z) z^{-2}\}$$

$$H_{ZT}(z) = \frac{Y_{ZT}(z)}{X_{ZT}(z)} = \frac{1}{2}\{1 + z^{-2}\}$$

$$H(w) = H_{ZT}(z)\Big|_{z=e^{jw}} = \frac{1}{2}\{1 + e^{-j2w}\} = e^{-jw}\cos(w)$$

Method - 4

$$y[n] \overset{DTFT}{\longleftrightarrow} Y(w) = \frac{1}{2}\{X(w) + X(w)e^{-j2w}\}.$$

$$H(w) = \frac{Y(w)}{X(w)} = \frac{1}{2}\{1 + e^{-j2w}\} = e^{-jw}\cos(w)$$

c) $|H(w)| = |e^{-jw}\cos(w)| = |e^{-jw}|\cdot|\cos(w)|$ — 2pt

$= |\cos(w)|$

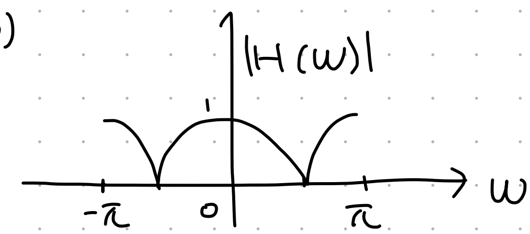
2pt : correct final value.

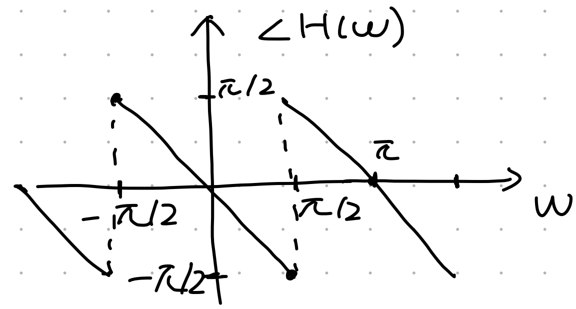
d) $\angle e^{-jw} = -w$ (1pt) $\quad \angle \cos(w) = \tan^{-1}\left(\frac{0}{\cos(w)}\right) = \begin{cases} 0, & \cos(w) > 0 \\ \pi / -\pi, & \cos(w) < 0 \end{cases}$ (1pt)

$$\angle H(w) = \angle e^{-jw} + \angle \cos(w) = \begin{cases} -w & \cos(w) > 0 \\ -w \pm \pi & \cos(w) < 0 \end{cases}$$

1pt: correct final value.

e)

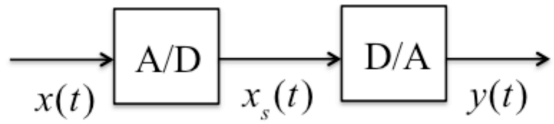




1.5pt: |H(w)| • x, y axis

1.5pt: ∠H(w) • correct plot.

2. (25 pts.) Consider the signal processing pipeline shown below:



To simplify matters, we will analyze this system entirely in the continuous-time domain. The sampling rate is 10 kHz. The input signal is $x(t) = \cos\left(2\pi(6000)t\right)$. At the output of the A/D convertor, we have $x_s(t) = \sum_{n=-\infty}^{\infty} x(n/10000)\delta(t - n/10000)$. At the output of the D/A convertor, we have $y(t) = x_s(t) * \mathrm{sinc}\left(10000t\right)$, where the $*$ denotes 1-D convolution.

a. (2) Find a simple closed-form expression for the CTFT $X(f)$ of $x(t)$. Your answer should not include any operators other than summations.

b. (7) Find a simple closed-form expression for the CTFT $X_s(f)$ of $x_s(t)$. Your answer should not include any operators other than summations.

c. (2) Sketch $X_s(f)$. Be sure to dimension all critical values in your sketch.

d. (7) Find a simple closed-form expression for the CTFT $Y(f)$ of $y(t)$. Your answer should not include any operators other than summations.

e. (2) Sketch $Y(f)$. Be sure to dimension all critical values in your sketch.

f. (5) Determine a simple expression for $y(t)$.

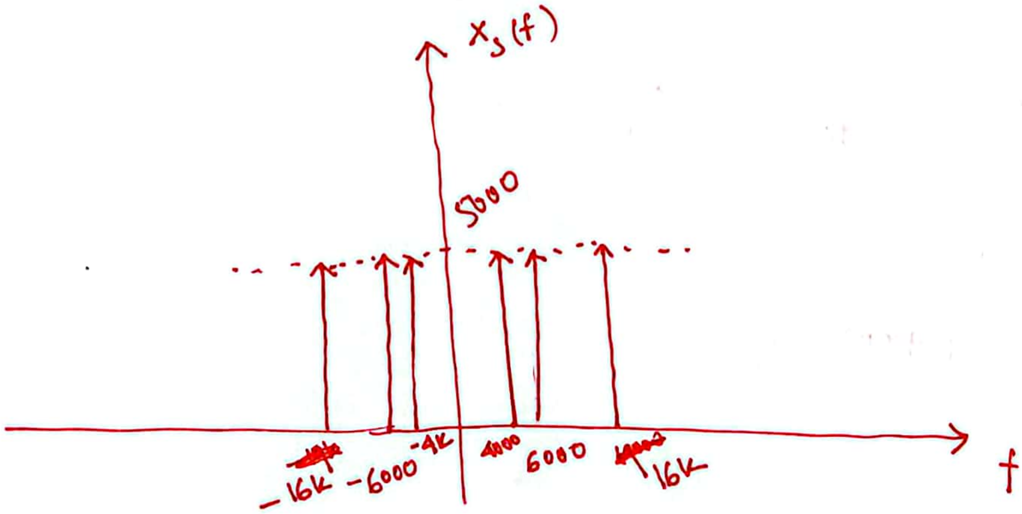
Final Exam

Ans:2

(a)
$$X(f) = F.T\{x(t)\}$$
$$= F.T\{\cos(2\pi(6000)t)\}$$
$$= \frac{1}{2}\left[\delta(f-6000) + \delta(f+6000)\right]$$

(b)
$$X_s(f) = F_s \sum_{k=-\infty}^{\infty} X(f - kF_s)$$
$$= 10000 \times \frac{1}{2} \sum_{k=-\infty}^{\infty} \left[\delta(f-6000-10000k) + \delta(f+6000-10000k)\right]$$
$$= 5000 \sum_{k=-\infty}^{\infty} \left[\delta(f-6000-10000k) + \delta(f+6000-10000k)\right]$$

(c)



(d) We know,

$\text{rect}(t) \xleftrightarrow{CTFT} \text{sinc}(f)$

Using reciprocity,

$\text{sinc}(t) \longleftrightarrow \text{rect}(-f)$

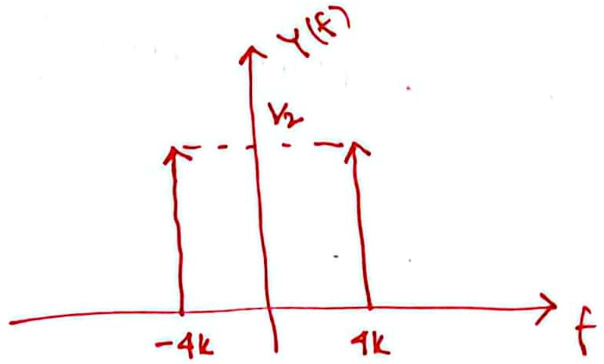
$\therefore \text{sinc}(1000t) \xleftrightarrow{CTFT} \frac{1}{10k}\text{rect}\left(\frac{f}{10000}\right) = W(f)$

$y(t) = x_s(t) * \text{sinc}(10000t)$

$\Rightarrow Y(f) = X_s(f)\, W(f)$

$= \frac{1}{2}\left[\delta(f-4000) + \delta(f+4000)\right]$

(e)



(f) $y(t) = \cos(2\pi(4000)t)$

3. (25 pts) In class, we analyzed linear predictive coding. The prediction is performed frame-by-frame on the speech signal; and the predictor has the form

$$\tilde{s}_n(m) = \sum_{k=1}^{p} \alpha_k s_n(m-k) ,$$

where n denotes the starting point (time index) of the frame (assuming a causal window), m denotes the time at which the prediction is being performed; and the α_k's, $k = 1,...,p$ denote the p predictor coefficients. We showed in class that according to the autocorrelation method, the predictor coefficients that minimize the total squared prediction error

$$E_n = \sum_{m=-\infty}^{\infty} \left(\tilde{s}_n(m) - s_n(m)\right)^2$$

for the frame starting at time n satisfy

$$R_n(l) = \sum_{k=1}^{p} \alpha_k R_n(k-l), \quad 1 \le l \le p, \tag{1}$$

where $R_n(k) = \sum_{m=-\infty}^{\infty} s_n(m) s_n(m+k)$ is the autocorrelation of the frame of speech data.

Show that if the predictor coefficients α_k, $k = 1,...,p$ satisfy Equation (1) above, then the prediction error simplifies to

$$E_n = R_n(0) - \sum_{k=1}^{p} \alpha_k R_n(k).$$

Problem 3 Solution

For the predictor $\tilde{s}_n(m) = \sum_{k=1}^{p} \alpha_k S_n(m-k)$ of the signal $S_n(m)$, the squared prediction error can be expanded as

$$E_n = \sum_{m=-\infty}^{\infty} \left(\tilde{S}_n(m) - S_n(m)\right)^2$$

$$E_n = \sum_{m=-\infty}^{\infty} \left(\sum_{k=1}^{p} \alpha_k S_n(m-k) - S_n(m)\right)^2$$

$$E_n = \sum_{m=-\infty}^{\infty} \left[S_n(m)S_n(m) + \left(\sum_{k=1}^{p} \alpha_k S_n(m-k)\right)^2 - 2S_n(m)\sum_{k=1}^{p} \alpha_k S_n(m-k)\right]$$

Squaring the summation over k:

$$E_n = \sum_{m=-\infty}^{\infty} S_n(m)S_n(m) + \sum_{m=-\infty}^{\infty}\sum_{k=1}^{p} \alpha_k S_n(m-k)\sum_{l=1}^{p} \alpha_l S_n(m-l) - 2\sum_{m=-\infty}^{\infty} S_n(m)\sum_{k=1}^{p} \alpha_k S_n(m-k)$$

Notice that the first term $\sum_{m=-\infty}^{\infty} S_n(m)S_n(m) = \sum_{m=-\infty}^{\infty} S_n(m)S_n(m+0) = R_n(0)$ is the autocorrelation of the frame of speech data for lag=0.

Then we switch the order of summation operators for the squared prediction error:

$$E_n = R_n(0) + \sum_{k=1}^{p}\sum_{l=1}^{p} \alpha_k \alpha_l \sum_{m=-\infty}^{\infty} S_n(m-k)S_n(m-l) - 2\sum_{k=1}^{p} \alpha_k \sum_{m=-\infty}^{\infty} S_n(m)S_n(m-k)$$

Identifying the summations that are $R_n(\cdot)$:

$$\sum_{m=-\infty}^{\infty} S_n(m-k)S_n(m-l) = R_n(k-l)$$

$$\sum_{m=-\infty}^{\infty} S_n(m)S_n(m-k) = R_n(k)$$

Then,

$$E_n = R_n(0) + \sum_{l=1}^{p} \alpha_l \sum_{k=1}^{p} \alpha_k R_n(k-l) - 2\sum_{k=1}^{p} \alpha_k R_n(k)$$

Given that the predictor coefficients $\alpha_k, k = 1, \dots, p$, that minimize the total squared prediction error satisfy $R_n(l) = \sum_{k=1}^{p} \alpha_k R_n(k-l)$

$$E_n = R_n(0) + \sum_{l=1}^{p} \alpha_l R_n(l) - 2\sum_{k=1}^{p} \alpha_k R_n(k)$$

$$E_n = R_n(0) + \sum_{k=1}^{p} \alpha_k R_n(k) - 2\sum_{k=1}^{p} \alpha_k R_n(k)$$

$$E_n = R_n(0) - \sum_{k=1}^{p} \alpha_k R_n(k)$$

Expanding the square (E_n): 1 pts

Squaring a summation (use different notation for two summations): 3 pts

Identifying $R_n(0)$: 3 pts

Knowing autocorrelation function is an even function: 3 pts

Switching the order of summation operators and identifying $R_n(k-l)$ and $R_n(k)$: 2 x 5 pts

Identifying and substituting $\sum_{k=1}^{p} \alpha_k R_n(k-l)$ by $R_n(l)$: 3 pts

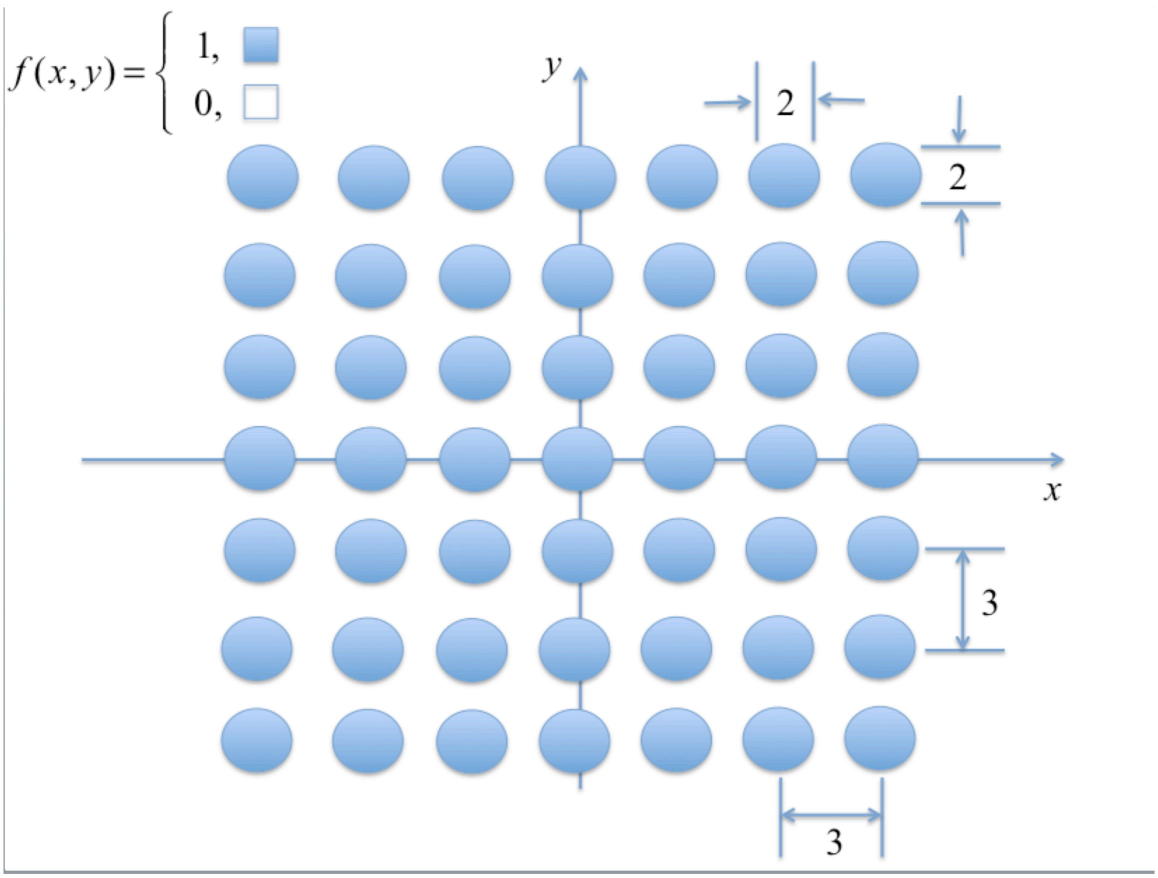
Other derivation steps: 2 pt

4. (25 pts.) For the function $f(x, y)$ given below, do the following: (Note that the picture shows all of $f(x, y)$.)

   a. (10) Express $f(x, y)$ in terms of the special functions given in class.

   b. (10) Find its CSFT $F(u, v)$ using transform pairs and properties. Your answer should not contain any operators other than summations. It also should not contain the convolution sign.

   c. (5) Sketch $F(u, v)$ in enough detail to show that you know what it looks like. Be sure to indicate the dimensions of all important quantities, and draw guidelines to indicate the shape of important features.



(a) Assume that

$$\text{(4 pts) } g(x,y) = \mathbf{rep}_{3,3}\big[\mathbf{circ}\big(\frac{x}{2}, \frac{y}{2}\big)\big]$$

Then,

$$\begin{aligned} f(x,y) &= \mathbf{rect}\big(\frac{x}{20}, \frac{y}{20}\big) \text{ (4 pts) } \cdot g(x,y) \\ &= \mathbf{rect}\big(\frac{x}{20}, \frac{y}{20}\big) \cdot \mathbf{rep}_{3,3}\big[\mathbf{circ}\big(\frac{x}{2}, \frac{y}{2}\big)\big] \text{ (2 pts)} \end{aligned}$$

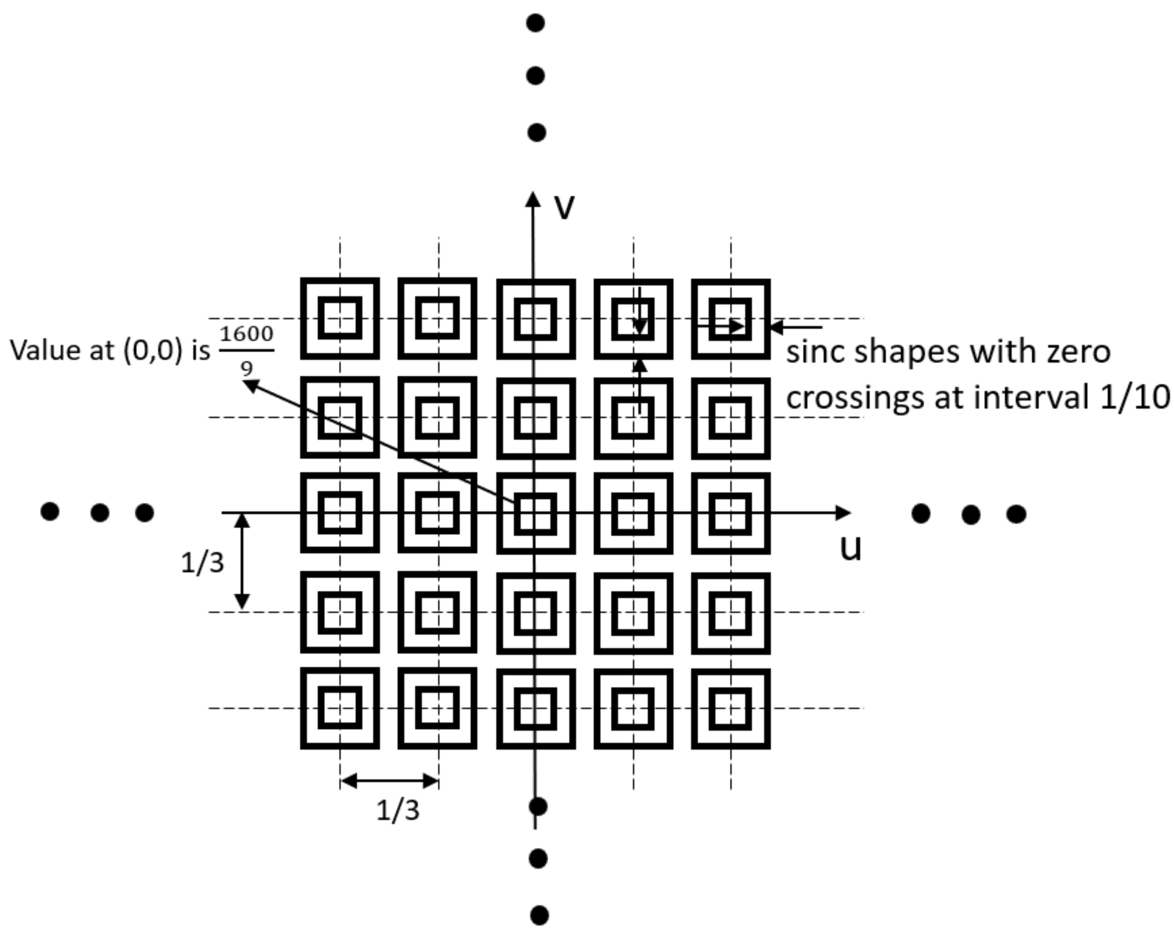
(b) DTFT of $g(x,y)$:

$$\text{(4 pts) } G(u,v) = \frac{1}{9}\,\mathbf{comb}_{\frac{1}{3},\frac{1}{3}}[4\,\mathbf{jinc}(2u,2v)]$$

Then, DTFT of $f(x,y)$:

$$\begin{aligned} F(u,v) &= 400\,\mathbf{sinc}(20u,20v) \text{ (4 pts) } **\, G(u,v) \\ &= \frac{400}{9}\,\mathbf{sinc}(20u,20v) ** \mathbf{comb}_{\frac{1}{3},\frac{1}{3}}[4\mathbf{jinc}(2u,2v)] \\ &= \frac{1600}{9} \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \mathbf{jinc}\big(\frac{2k}{3}, \frac{2l}{3}\big)\mathbf{sinc}\big[10(u-\frac{k}{3}), 10(v-\frac{l}{3})\big] \text{ (2 pts)} \end{aligned}$$

(c) The plot is attached below.



The magnification of sinc is determined by the jinc function values.
Jinc roll-off with first zero crossing at approximately 3/2.

(1 pt) Shape of the spectrum. (1 pt) Sinc shape. (1 pt) Jinc shape. (1 pt) u and v axis labels and unit. (1 pt) Quantities.

5. (25 pts) Consider a spatial filter with point spread function $h[m,n]$ given below

$h[m,n]$		$n=-1$	$n=0$	$n=1$
m	-1	$-\frac{1}{9}$	$-\frac{1}{9}$	$-\frac{1}{9}$
	0	$-\frac{1}{9}$	$\frac{17}{9}$	$-\frac{1}{9}$
	1	$-\frac{1}{9}$	$-\frac{1}{9}$	$-\frac{1}{9}$

a. (9) Find the output $g[m,n]$ when this filter is applied to the following 9×9 input image. You may assume that the boundary pixel values are extended beyond the boundary. You need only calculate the output over the original 9×9 set of pixels in the input image.

1	0	0	0	0	0	0	0	0
1	1	0	0	0	0	0	0	0
1	1	1	0	0	0	0	0	0
1	1	1	1	0	0	0	0	0
1	1	1	1	1	0	0	0	0
1	1	1	1	1	1	0	0	0
1	1	1	1	1	1	1	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0

b. (12) Find a simple expression for the frequency response $H(\mu,\nu)$ of this filter, and sketch the magnitude $|H(\mu,\nu)|$ along the μ axis ($\nu = 0$), the ν axis ($\mu = 0$), the $\mu = \nu$, and the $\mu = -\nu$ axes.

c. (3) Using your results from parts (a) and (b), explain what this filter does. Relate spatial domain properties to frequency domain properties. Be sure to examine what happens at each edge of the region of 1's above, and how this relates to the frequency domain, as well as what happens in the center of the region of 1's and in the border of 0's that surrounds the non-zero portion of the image.

d. (1) Is the filter DC-preserving? Why or why not?

(a5)

$$g(m,n) = \frac{1}{9}\begin{bmatrix} 11 & -4 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 10 & 12 & -3 & -1 & 0 & 0 & 0 & 0 & 0 \\ 9 & 10 & 12 & -3 & -1 & 0 & 0 & 0 & 0 \\ 9 & 9 & 10 & 12 & -3 & -1 & 0 & 0 & 0 \\ 9 & 9 & 9 & 10 & 12 & -3 & -1 & 0 & 0 \\ 9 & 9 & 9 & 9 & 10 & 12 & -3 & -1 & 0 \\ 12 & 12 & 12 & 12 & 12 & 13 & -3 & -1 & 0 \\ -3 & -3 & -3 & -3 & -3 & -3 & -2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(b)

$$h[m,n] = \begin{bmatrix} -\frac{1}{9} & -\frac{1}{9} & -\frac{1}{9} \\ -\frac{1}{9} & \frac{17}{9} & -\frac{1}{9} \\ -\frac{1}{9} & -\frac{1}{9} & -\frac{1}{9} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \left(\begin{bmatrix} \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \\ \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \\ \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \end{bmatrix} \right)$$

$h_1[m]$: $\begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}$

$$\begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

$-1 \quad 0 \quad 1$

$h_2(n)$.

$$= 2\delta[k,l] - h_1(m)\, h_2(n)$$

$$= 2\delta[k,l] - \frac{1}{9}\left[\delta(m+1) + \delta(m) + \delta(m-1)\right]$$

$$\left[\delta(n+1) + \delta(n) + \delta(n-1)\right]$$

$$H_1(u) = \frac{1}{3} e^{ju} + \frac{1}{3} + \frac{1}{3} e^{-ju}$$

$$= \frac{1}{3}\left(1 + e^{ju} + e^{-ju}\right)$$

$$= \frac{1}{3}\left(1 + 2\cos u\right)$$

$$H_2(v) = \frac{1}{3} e^{jv} + \frac{1}{3} + \frac{1}{3} e^{-jv}$$

$$= \frac{1}{3}\left(1 + e^{jv} + e^{-jv}\right)$$

$$= \frac{1}{3}\left(1 + 2\cos v\right)$$

$$H(u,v) = 2 - \frac{1}{9}(1 + 2\cos u)(1 + 2\cos v)$$

For $v = 0$, along u-axis.

$$H(u,0) = 2 - \frac{1}{9}(1 + 2\cos u)(3)$$

$$= 2 - \frac{1}{3}(1 + 2\cos u)$$

For $u = 0$, along v-axis,

$$H(0,v) = 2 - \frac{1}{9} \times 3 \cdot (1 + 2\cos v)$$

$$= 2 - \frac{1}{3}(1 + 2\cos v)$$

for $\mu = v$,

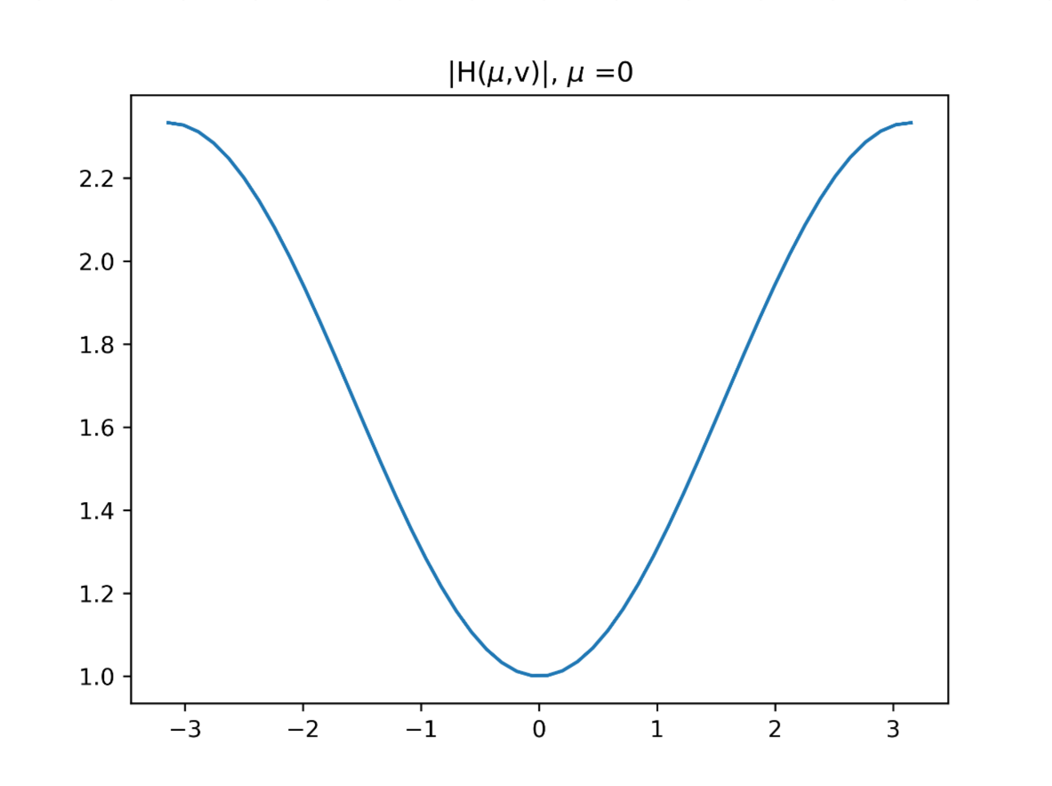
$$H(\mu,\mu) = 2 - \frac{1}{9}(1+2\cos\mu)^2$$

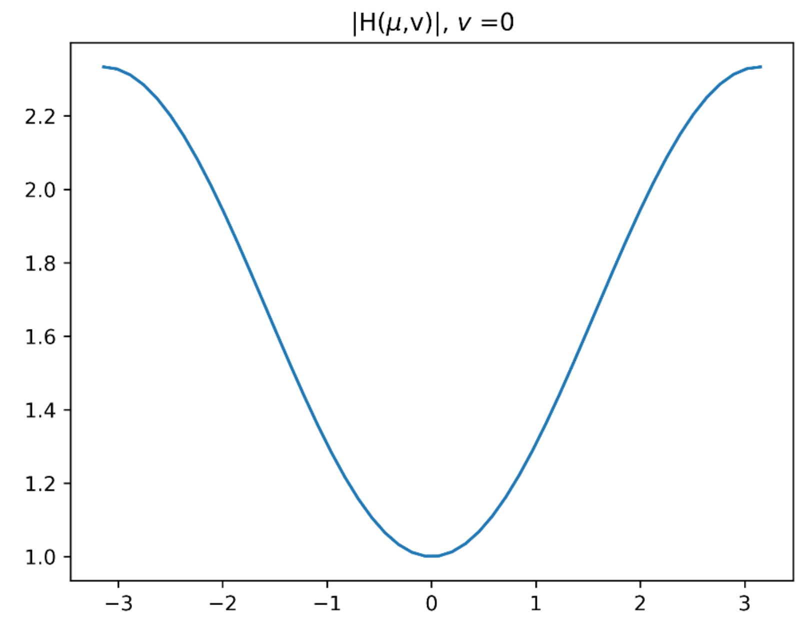
for $\mu = -v$

$$H(\mu,-v) = 2 - \frac{1}{9}(1+2\cos\mu)(1+2\cos(-\mu))$$

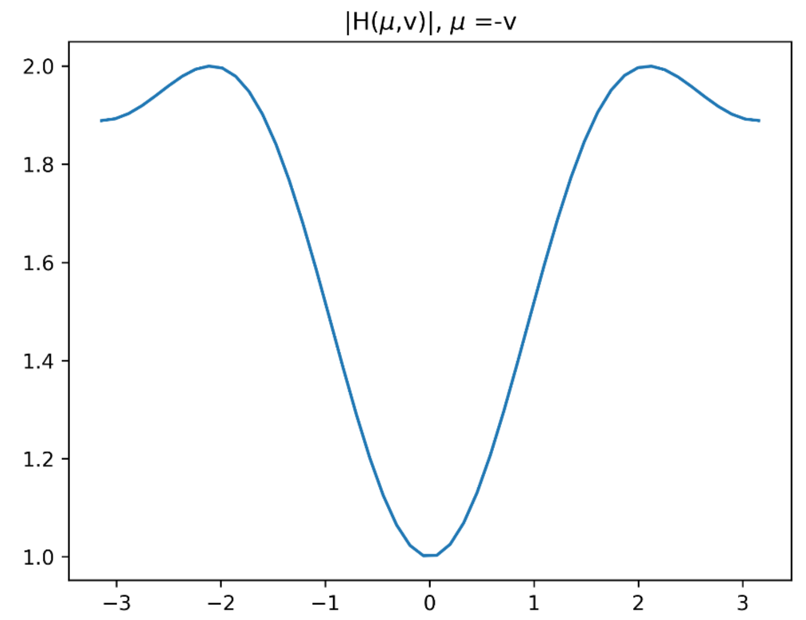
$$= 2 - \frac{1}{9}(1+2\cos\mu)(1+2\cos\mu)$$

$$= 2 - \frac{1}{9}(1+2\cos\mu)^2$$

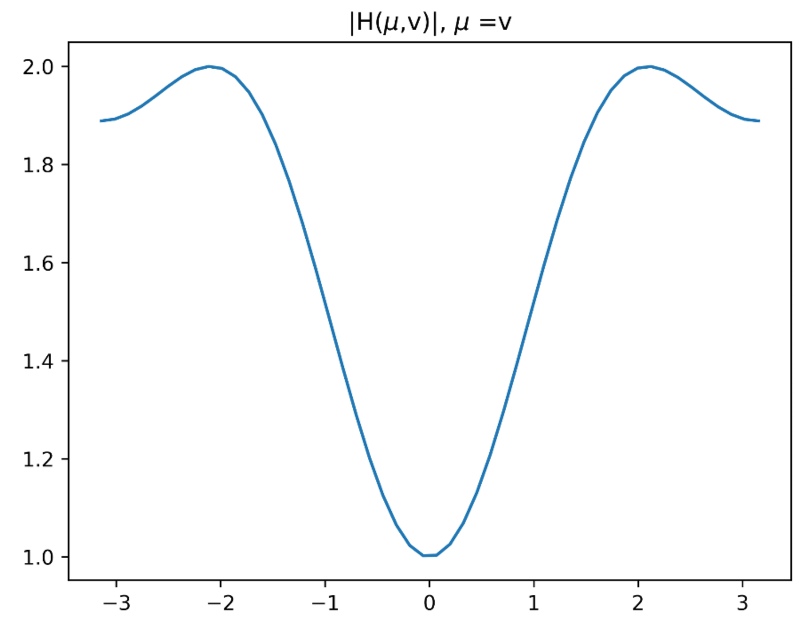




|H(μ,v)|, v =0
2.2
2.0
1.8
1.6
1.4
1.2
1.0
−3
−2
−1
0
1
2
3



|H(μ,v)|, μ =-v
2.0
1.8
1.6
1.4
1.2
1.0
−3
−2
−1
0
1
2
3



|H(μ,v)|, μ =v
2.0
1.8
1.6
1.4
1.2
1.0
−3
−2
−1
0
1
2
3

(c) The center region of 1's is preserved,

→ since this filter is DC preserving

→ The filter is a high pass filter along both a u & v - axes & thus preserves all the edges of 1's which can be observed from $g[m,n]$

→ Along the $u=v$ & $u=-v$ the filter accentuate the output first & then dampens it which is visible from the diagonal elements.

(d) Yes it is DC preserving because,

$$\sum_m \sum_n h(m,n) = 1.$$