19 April 2023 **Name:_____**

ECE 438 **Exam No. 3** **Spring 2023**

- You have 50 minutes to work the following four problems.
- Be sure to show all your work to obtain full credit.
- The exam is closed book and closed notes.
- Calculators, smart phones, and smart watches are not permitted, and must be put away.

1. (25 pts.) Consider a random sequence $V_n, n = \ldots -2,-1,0,1,2\ldots$, where each sample in the sequence is a random variable with zero mean. Suppose we quantize these random variables individually. Using an optimum nonlinear quantizer, we will achieve a certain root-mean-squared quantization error, which we would expect will decrease with decreasing σ_V^2. However, if the random variables are highly correlated, we might expect to achieve a lower root-mean-squared quantization error by quantizing the difference between successive random variables, rather than quantizing them individually.

   The goal of this problem is to explore this idea. To keep things as simple as possible, we will only consider two random variables X and Y, which could be successive samples from the sequence V_n discussed above. To keep things really simple, we will assume that $\overline{X} = \overline{Y} = 0$, $\overline{X^2} = \overline{Y^2}$, and that X and Y have correlation coefficient ρ_{XY}. Then we define a third random variable $Z = X - Y$.

   a. (5) Find the expected value $\overline{Z} = E\{Z\}$.

   b. (15) Find the second moment $\overline{Z^2} = E\{Z^2\}$ in terms of $\overline{X^2}$ and ρ_{XY}.

   c. (3) Discuss how $\overline{Z^2}$ varies as ρ_{XY} approaches 1, and as ρ_{XY} approaches 0.

   d. (2) Discuss the implications of the dependence of $\overline{Z^2}$ on ρ_{XY} for quantizing the difference between successive random variables in the sequence $V_n, n = \ldots -2,-1,0,1,2\ldots$.

1. (25 pts.) Consider a random sequence V_n , $n = ...-2,-1,0,1,2...$, where each sample in the sequence is a random variable with zero mean. Suppose we quantize these random variables individually. Using an optimum nonlinear quantizer, we will achieve a certain root-mean-squared quantization error, which we would expect will decrease with decreasing σ_V^2. However, if the random variables are highly correlated, we might expect to achieve a lower root-mean-squared quantization error by quantizing the difference between successive random variables, rather than quantizing them individually.

   The goal of this problem is to explore this idea. To keep things as simple as possible, we will only consider two random variables X and Y , which could be successive samples from the sequence V_n discussed above. To keep things really simple, we will assume that $\overline{X} = \overline{Y} = 0, \overline{X^2} = \overline{Y^2}$, and that X and Y have correlation coefficient ρ_{XY}. Then we define a third random variable $Z = X - Y$.

   a. (5) Find the expected value $\overline{Z} = E\{Z\}$.

   b. (15) Find the second moment in $\overline{Z^2} = E\{Z^2\}$ in terms of $\overline{X^2}$ and ρ_{XY} .

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   d. (2) Discuss the implications of the dependence of $\overline{Z^2}$ on ρ_{XY} for quantizing the difference between successive random variables in the sequence V_n , n = ...−2,−1,0,1,2….

Solution:

a.

$$\begin{aligned}\overline{Z} &= E\{Z\} = E\{X - Y\} \\ &= E\{X\} - E\{Y\} && \text{(linearity of expectation)} \\ &= 0\end{aligned}$$

b.

$$\begin{aligned}\overline{Z^2} &= E\{Z^2\} \\ &= E\{(X - Y)(X - Y)\} \\ &= E\{X^2 + Y^2 - 2XY\} \\ &= E\{X^2\} + E\{Y^2\} - 2E\{XY\} && \text{(linearity of expectation)} \\ &= 2E\{X^2\} - 2E\{XY\} && (\overline{X^2} = \overline{Y^2})\end{aligned}$$

$$\sigma_{XY}^2 = E\{XY\} - E\{X\}E\{Y\}$$

$$\begin{aligned}E\{XY\} &= \sigma_{XY}^2 && (\overline{X} = \overline{Y} = 0) \\ &= \rho_{XY}\sigma_X\sigma_Y && (\rho_{XY} = \frac{\sigma_{XY}^2}{\sigma_X\sigma_Y})\end{aligned}$$

$$Var\{X\} = E\{X^2\} - (E\{X\})^2 = \overline{X^2}$$

$$Var\{Y\} = E\{Y^2\} - (E\{Y\})^2 = \overline{Y^2}$$

Therefore we have

$$\overline{Z^2} = 2E\{X^2\} - 2E\{XY\}$$

$$= 2E\{X^2\} - 2\rho_{XY}\sigma_X\sigma_Y$$

$$= 2\overline{X^2} - 2\rho_{XY}\sqrt{\overline{X^2Y^2}}$$

$$= 2\overline{X^2} - 2\rho_{XY}\overline{X^2}$$

$$= 2(1 - \rho_{XY})\overline{X^2}$$

c. As ρ_{XY} approaches 1, $\overline{Z^2}$ approaches 0; as ρ_{XY} approaches 0, $\overline{Z^2}$ approaches $2\overline{X^2}$.

d. The dependence of $\overline{Z^2}$ on ρ_{XY} implies that we might achieve a lower root-mean-squared quantization error by quantizing the difference between successive random variables in the sequence V_n. This is because these variables can be highly correlated and have a correlation coefficient closer to 1.

2. (25 pts.) Consider a wide-sense stationary discrete-time random sequence $X_n, n = \ldots,-2,-1,0,1,2,\ldots$ with mean 0.5 and autocorrelation function as stated below:

$$r_{XX}[n] = \begin{cases} 1, & n = 0 \\ \frac{1}{2}, & n = \pm 1 \\ 0, & \text{else} \end{cases} .$$

The sequence X_n is input to a filter with output Y_n. The filter equation is given by:

$$Y_n = \tfrac{1}{2}\left\{X_n + X_{n-1}\right\}$$

a. (5) Find the expected value $\overline{Y}_n = E\left\{Y_n\right\}$ of the output.

b. (10) Find the cross-correlation $r_{XY}\left[m,n\right] = E\left\{X_m Y_n\right\}$ between the input and the output.

c. (9) Find the autocorrelation $r_{YY}\left[m,n\right] = E\left\{Y_m Y_n\right\}$ of the output.

d. (1) Is the output sequence wide-sense stationary?

Hint: Use the formulas on p. 6 of the formulas document.

ECE 438 Exam 3 Problem 2

(a)

$$
\begin{aligned}
E[Y_n] =& E[\frac{1}{2}X_n + X_{n-1}] \\
\text{(1 pts)} \quad =& \frac{1}{2}E[X_n] + \frac{1}{2}E[X_{n-1}] \\
\text{(1 pts)} \quad =& \frac{1}{4} + \frac{1}{4}
\end{aligned}
$$

(1 pts) Since X_n is w.s.s, X_n has constant mean value and $E[X_n]$ independent of n

$$
\text{(2 pts)} \quad = \frac{1}{2}
$$

(b) Method 1:

$$
\begin{aligned}
\text{(2 pts)} \; r_{XY}[m,n] =& E\big[X[m]Y[m+n]\big] \\
=& E\big[X[m](\frac{1}{2}\{X[m+n] + X[m+n-1]\})\big] \\
=& \frac{1}{2}E\big[X[m]X[m+n]\big] + \frac{1}{2}E\big[X[m]X[m+n-1]\big] \\
\text{(2 pts)} \quad =& \frac{1}{2}r_{XX}[n] + \frac{1}{2}r_{XX}[n-1] \\
\text{(2 pts)} \quad =& \frac{1}{2}\Big(\delta[n] + \frac{1}{2}\delta[n-1] + \frac{1}{2}\delta[n+1]\Big) \\
\text{(2 pts)} \quad & + \frac{1}{2}\Big(\delta[n-1] + \frac{1}{2}\delta[n-2] + \frac{1}{2}\delta[n]\Big) \\
\text{(2 pts)} \quad =& \frac{1}{4}\delta[n+1] + \frac{3}{4}\delta[n] + \frac{3}{4}\delta[n-1] + \frac{1}{4}\delta[n-2]
\end{aligned}
$$

Method 2:

$$
\begin{aligned}
\text{(2 pts)} \; r_{XY}[m,n] =& r_{XX}[n] * h[n] \\
=& \Big(\delta[n] + \frac{1}{2}\delta[n-1] + \frac{1}{2}\delta[n+1]\Big) \text{ (2 pts) } * \Big(\frac{1}{2}(\delta[n] + \delta[n-1])\Big) \text{ (2 pts)} \\
\text{(2 pts)} \quad =& \frac{1}{2}\Big(\delta[n] + \delta[n-1] + \frac{1}{2}\delta[n-1] + \frac{1}{2}\delta[n-2] + \frac{1}{2}\delta[n+1] + \frac{1}{2}\delta[n]\Big) \\
\text{(2 pts)} \quad =& \frac{1}{4}\delta[n+1] + \frac{3}{4}\delta[n] + \frac{3}{4}\delta[n-1] + \frac{1}{4}\delta[n-2]
\end{aligned}
$$

(c) Method 1:

$$
\begin{aligned}
\text{(2 pts) } r_{YY} =& E\big[Y[m]Y[m+n]\big] \\
=& E\Big[\frac{1}{2}\big(X[m]+X[m-1]\big)\frac{1}{2}\big(X[m+n]+X[m+n-1]\big)\Big] \\
=& \frac{1}{4}\Big(E\big[X[m]X[m+n]\big]+E\big[X[m]X[m+n-1]\big] \\
& +E\big[X[m-1]X[m+n]\big]+E\big[X[m-1]X[m+n-1]\big]\Big) \\
\text{(2 pts) } =& \frac{1}{4}\Big(r_{XX}[n]+r_{XX}[n-1]+r_{XX}[n+1]+r_{XX}[n]\Big) \\
\text{(2 pts) } =& \frac{1}{8}\delta[n+2]+\frac{1}{2}\delta[n+1]+\frac{3}{4}\delta[n]+\frac{1}{2}\delta[n-1]+\frac{1}{8}\delta[n-2]
\end{aligned}
$$

If all steps are correct, 3 pts

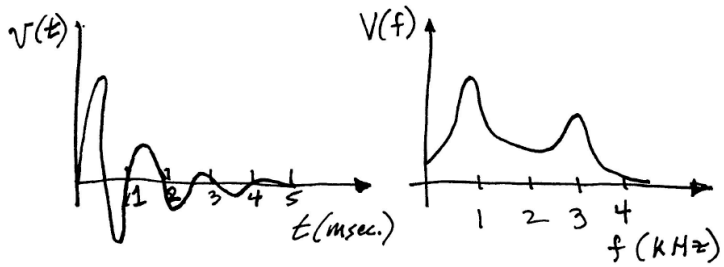
Method 2:

$$
\begin{aligned}
\text{(2 pts) } r_{YY}[n] =& h[n]*r_{XY}[-n] \\
=& \Big(\frac{1}{2}\big(\delta[n]+\delta[n-1]\big)\Big)*\Big(\frac{1}{4}\delta[-n+1]+\frac{3}{4}\delta[-n]+\frac{3}{4}\delta[-n-1]+\frac{1}{4}\delta[-n-2]\Big) \\
=& \Big(\frac{1}{2}\big(\delta[n]+\delta[n-1]\big)\Big)*\Big(\frac{1}{4}\delta[n-1]+\frac{3}{4}\delta[n]+\frac{3}{4}\delta[n+1]+\frac{1}{4}\delta[n+2]\Big) \\
\text{(2 pts) } =& \frac{1}{8}\delta[n-1]+\frac{3}{8}\delta[n]+\frac{3}{8}\delta[n+1]+\frac{1}{8}\delta[n+2] \\
& +\frac{1}{8}\delta[n-2]+\frac{3}{8}\delta[n-1]+\frac{3}{8}\delta[n]+\frac{1}{8}\delta[n+1] \\
\text{(2 pts) } =& \frac{1}{8}\delta[n+2]+\frac{1}{2}\delta[n+1]+\frac{3}{4}\delta[n]+\frac{1}{2}\delta[n-1]+\frac{1}{8}\delta[n-2]
\end{aligned}
$$

If all steps are correct, 3 pts

(d) (1 pts) **YES**. The output sequence Y_n **IS** wide-sense stationary.
Because it has constant mean of $\frac{1}{2}$ and its autocorrelation r_{YY} is time-invariant.

3. (30) Consider a voiced phoneme for which the time-domain continuous-time vocal tract response $v(t)$ and corresponding frequency response (CTFT) $V(f)$ are given below.



a. (8) Assume that the pitch frequency for the speaker is 200 Hz. Sketch what the continuous-time domain speech waveform $s(t)$ would look like in this case. Be sure to dimension all important quantities in the speech waveform.

b. (12) Suppose that we sample the speech waveform $s(t)$ above at a 10 kHz rate, and compute the short-time discrete-time Fourier transform (STDTFT) using a window of length 25 samples. Carefully sketch the resulting spectrogram. Be sure to dimension all important quantities in your plot. Is this a wide-band or narrow-band spectrogram?

c. (10) Suppose that we sample the speech waveform $s(t)$ above at a 10 kHz rate, and compute the short-time discrete-time Fourier transform (STDTFT) using a window of length 1,000 samples. Carefully sketch the resulting spectrogram. Be sure to dimension all important quantities in your plot. Is this a wide-band or narrow-band spectrogram?

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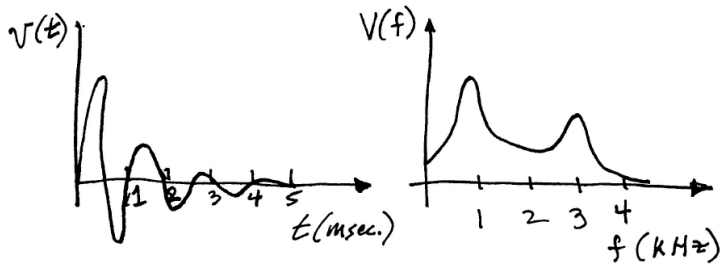

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c. (10) Suppose that we sample the speech waveform $s(t)$ above at a 10 kHz rate, and compute the short-time discrete-time Fourier transform (STDTFT) using a window of length 1,000 samples. Carefully sketch the resulting spectrogram. Be sure to dimension all important quantities in your plot. Is this a wide-band or narrow-band spectrogram?

Solution:

(a). Given, pitch frequency = 200 Hz.

So, pitch period, $P = \frac{1}{200}$ s.

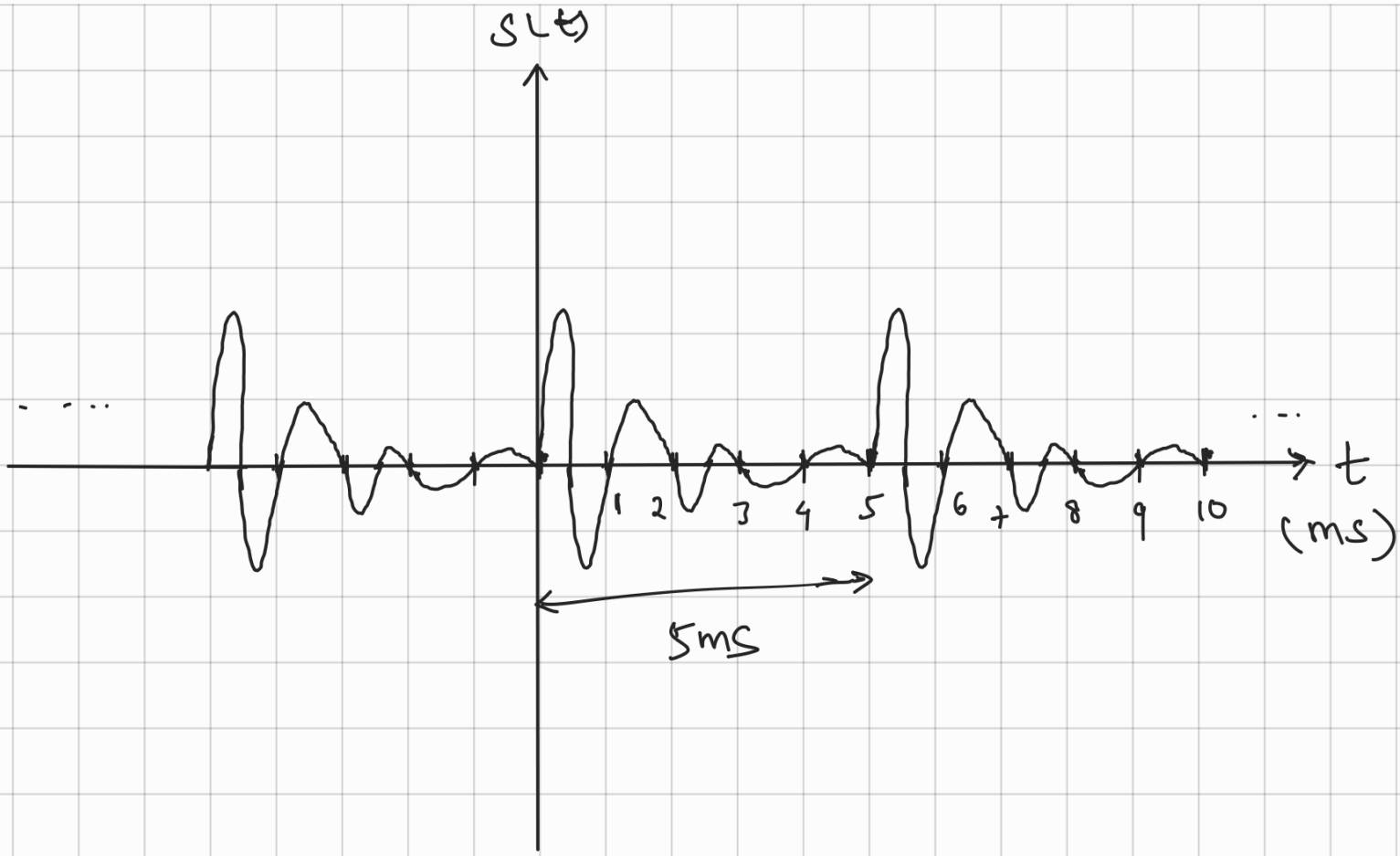
$= 0.005$ s

$= \underline{5 \text{ ms}}$.

So,

$$s(t) = \text{rep}_P \{v(t)\}$$

$$= \sum_{k=-\infty}^{\infty} v(t - 0.005k).$$



(b) Using V(f) we can say that the two formant frequencies are,

$F_1 = 1\,KHz$ & $F_2 = 3\,KHz$

No of samples in a pitch period.

$$= 5\times10^{-3}\,s \times \underbrace{10\times10^{3}}_{\text{sampling frequency}} \text{ samples/s.}$$

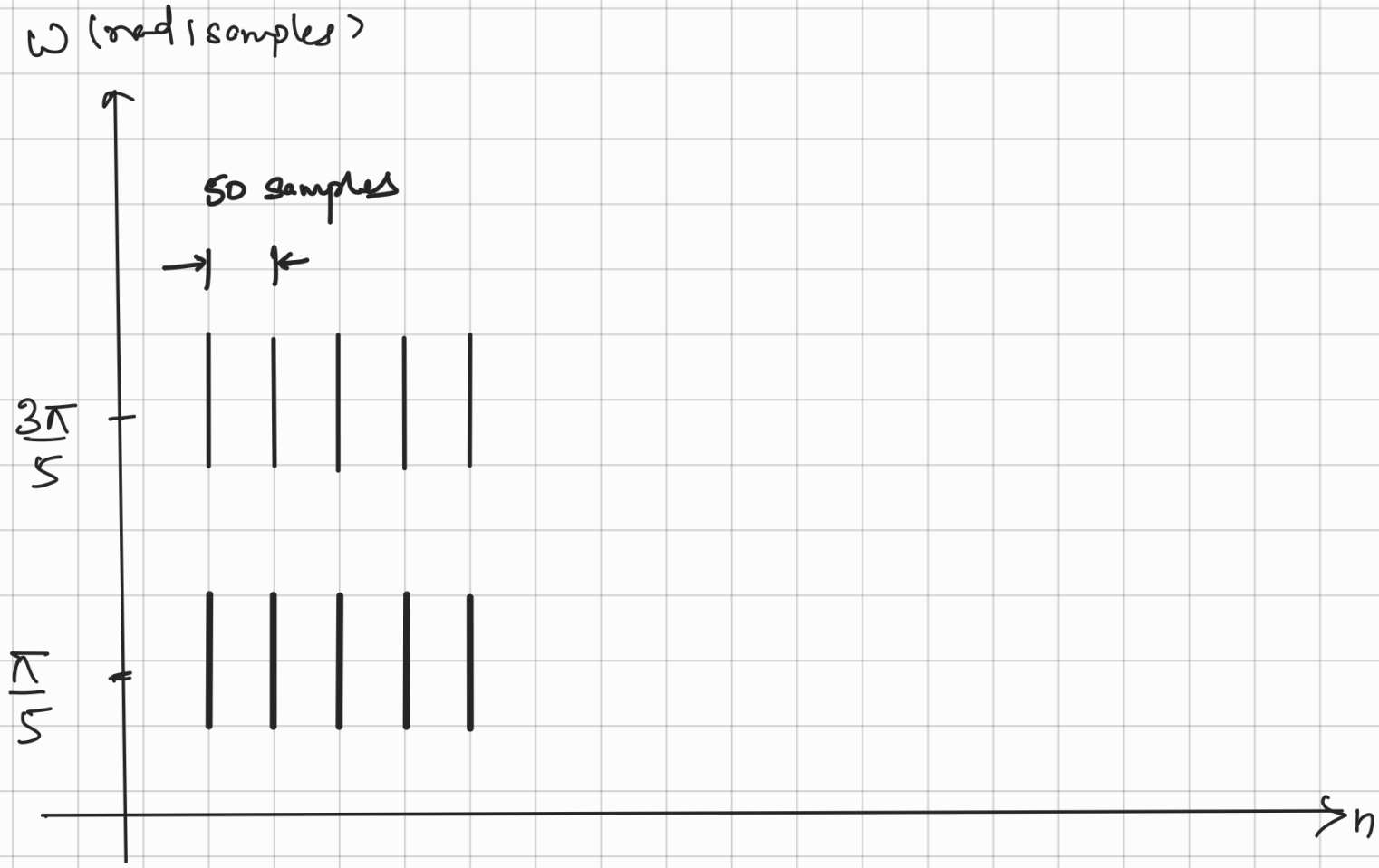
$$= 50 \text{ samples}$$

We have been given a window of length 25 samples.

$\therefore$ $\underline{50 > 25}$. So this is wide-band.

$$\omega_1 = F_1 \cdot \frac{2\pi}{f_s} = 1 \cdot \frac{2\pi}{10} = \frac{\pi}{5}$$

$$\omega_2 = F_2 \cdot \frac{2\pi}{f_s} = 3 \cdot \frac{2\pi}{10} = \frac{3\pi}{5}$$



(c) Again,

$F_1 = 1\,KHz$, $f_2 = 3KHz$

No of samples in pitch period = 50

Also, we have been given a window of length = 1000 samples.

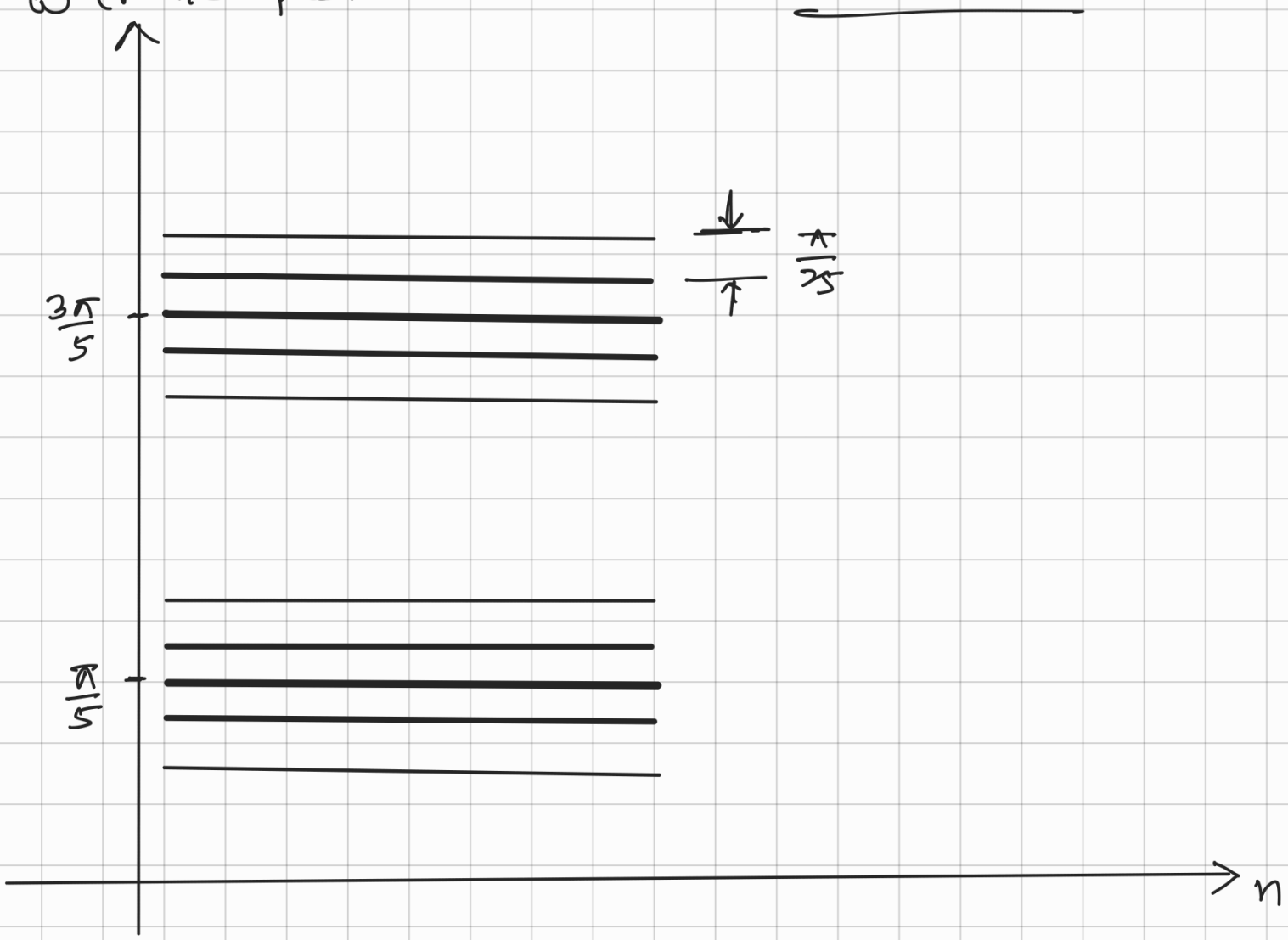
So, $50 < 1000$

$\Rightarrow$ This is narrowband.

$\omega_1 = \frac{\pi}{5}$, $\omega_2 = \frac{3\pi}{5}$

Pitch freq. (in rad/samples) $= 200 \times \frac{2\pi}{10 \times 10^3} = \frac{\pi}{25}$ rad/samples

(Not to scale)



4. (20 pts) Consider an L-channel modulated filter bank, where the filters have frequency response.

$$H_l(\omega) = H_0\left(\omega - \frac{2\pi l}{L}\right), 0 \le \omega \le 2\pi,\ l = 0,1,\ldots,L-1,$$

and $H_0(\omega) = \mathrm{rect}\left(\frac{L\omega}{2\pi}\right), -\pi \le \omega \le \pi$.

a. (4) Sketch $H_0(\omega)$. Be sure to dimension all important quantities.

b. (4) For the fixed value $L = 4$, show graphically that these filters satisfy the frequency domain condition for perfect reconstruction: $\sum_{l=0}^{L-1} H_l(\omega) \equiv 1, 0 \le \omega \le 2\pi$.

c. (8) Determine an expression for the unit sample response $h_0[n]$ of the baseband filter.

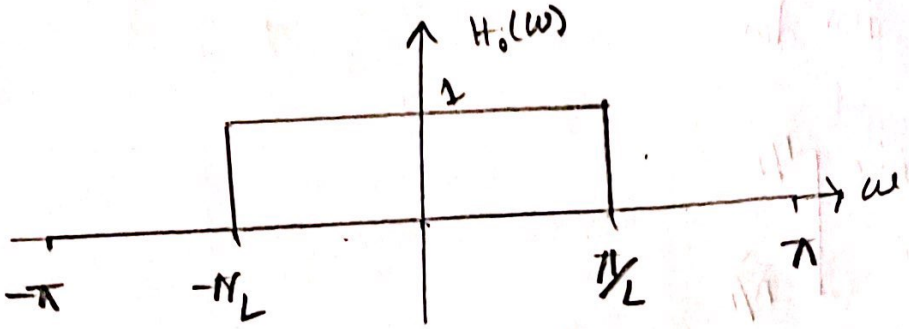
d. (4) Show that this filter satisfies the time-domain condition for perfect reconstruction:

$$h_0[n] = \begin{cases} \frac{1}{L}, & n = 0, \\ 0, & n = kL, k \ne 0, \\ \text{don't care}, & \text{else} \end{cases}$$

Exam-3

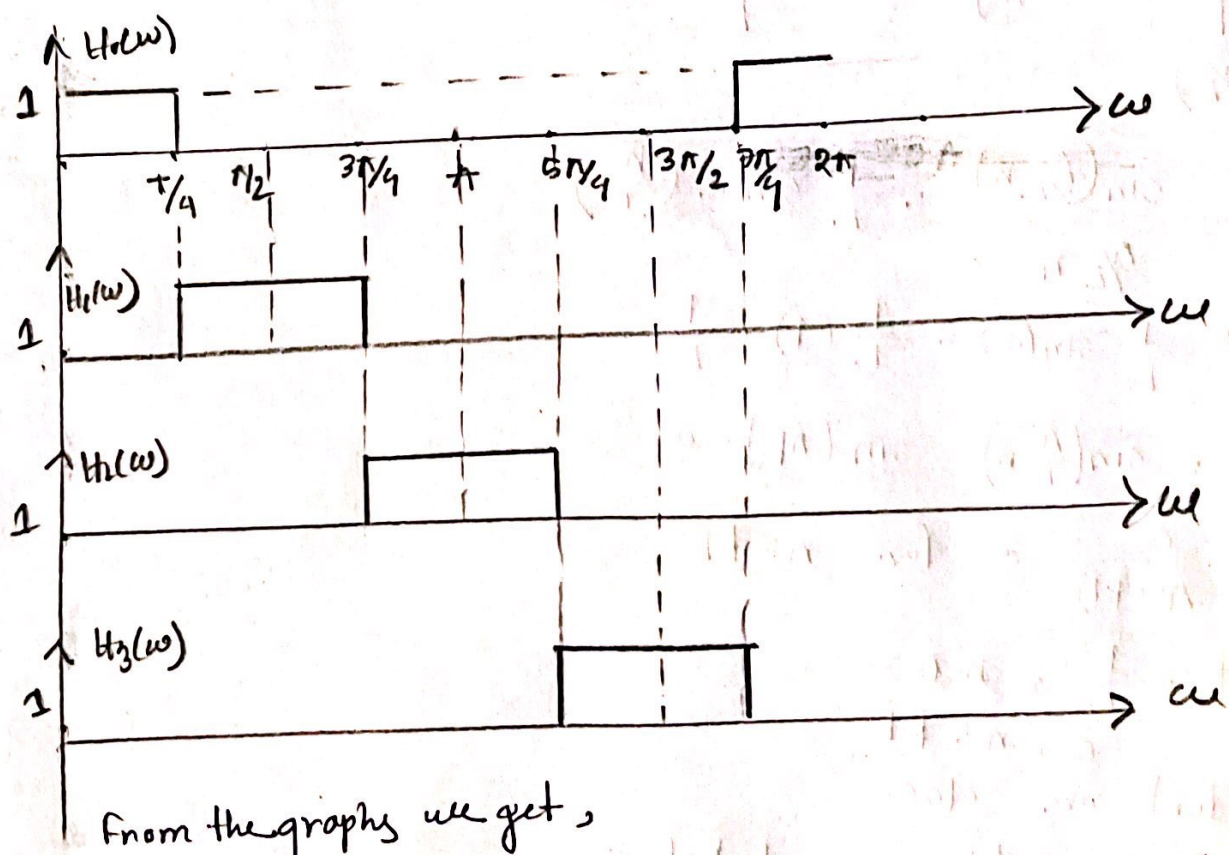
Problem-4

a. $H_0(\omega) = \text{rect}\left(\frac{L\omega}{2\pi}\right) = \text{rect}\left(\frac{\omega/2\pi}{L}\right) = \begin{cases} 1, & |\omega| \le \frac{\pi}{L} \\ 0, & \frac{\pi}{L} < |\omega| \le \pi \end{cases}$



b. Frequency domain cond^n :-

$$\sum_{l=0}^{L-1} H_l(\omega) = 1 \quad \forall \omega, \; 0 \le \omega \le 2\pi \quad (L=4)$$



From the graphs we get,

$H_0(\omega) + H_1(\omega) + H_2(\omega) + H_3(\omega) = 4, \; \forall \omega, \; 0 \le \omega \le 2\pi$

$\therefore$ Frequency domain condition is satisfied.

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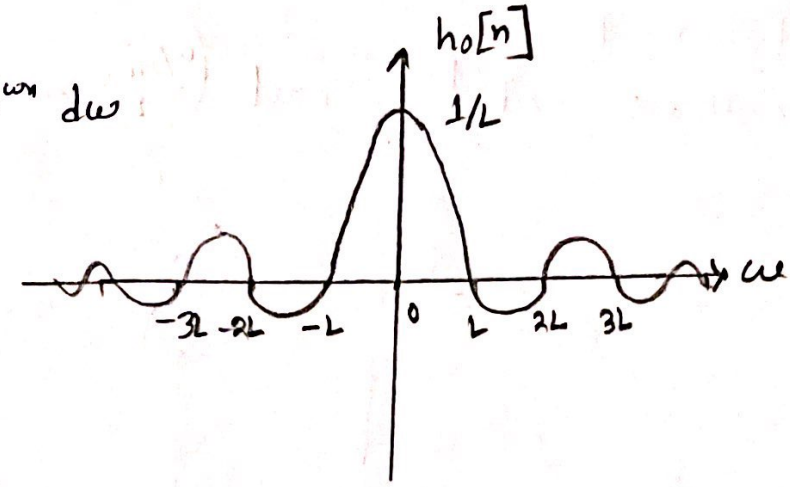
4 (c) $h_0[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_0(\omega)\, e^{j\omega n}\, d\omega$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \text{rect}\left(\omega/2\pi/L\right) e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi/L}^{\pi/L} e^{j\omega n}\, d\omega$$



$$= \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{-\pi/L}^{\pi/L}$$

$$= \frac{1}{2\pi n j}\left[e^{jn\pi/L} - e^{-jn\pi/L}\right]$$

$$= \frac{1}{\pi n}\left[\frac{e^{jn\pi/L} - e^{-jn\pi/L}}{2j}\right]$$

$$= \frac{\sin\left(n\pi/L\right)}{n\pi}$$

$$= \frac{1}{L}\,\frac{\sin\left(\pi/L\, n\right)}{\pi/L\, n} = \frac{1}{L}\,\text{sinc}\left(\frac{n}{L}\right)$$

(d) $\lim_{n\to 0} h_0[n] = \frac{1}{L}\sin(0) = \frac{1}{L}(1) = \frac{1}{L}$

for, $n = kL$, $\sin\left(\frac{\pi}{L}n\right) = \sin(\pi k) = 0$

$\therefore\ h_0[n] = \frac{0}{\pi(kL)} = 0$ for $n = kL$

$\therefore\ h_0[n] = \begin{cases} \frac{1}{L}, & n=0 \\ 0, & n = kL \\ \text{don't care}, & \text{else} \end{cases}$

$\therefore$ Time domain condition is satisfied.

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