22 March 2023 **Name:___________________**

ECE 438 **Exam No. 2** **Spring 2023**

- You have 50 minutes to work the following four problems.
- Be sure to show all your work to obtain full credit.
- The exam is closed book and closed notes.
- Calculators, smart phones, and smart watches are not permitted, and must be put away.

1. (25 pts) Consider the causal DT system with the following transfer function

$$H_{ZT}(z) = \frac{\left(1 - \frac{1}{2}z^{-1}\right)}{\left(1 + \frac{1}{2}z^{-1}\right)}$$

Suppose that the input to this system is given by

$$x[n] = \left(\frac{1}{3}\right)^n u[n].$$

a. (5) Find the ZT $X(z)$. Be sure to state the region of convergence.

b. (10) Find the partial fraction expansion for $Y(z)$.

c. (10) Based on your answer to part (b), find the output $y[n]$.

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Solution:

a. Using the transform pair

$$a^n u[n] \overset{ZT}{\leftrightarrow} \frac{1}{1 - az^{-1}}, |z| > |a|$$

We have

$$x(z) = \frac{1}{1 - \frac{1}{3}z^{-1}}, |z| > \frac{1}{3}$$

(3 pts: ZT result, 2 pts: ROC)

b.

$$Y(z) = H(z)X(z) = \frac{\left(1 - \frac{1}{2}z^{-1}\right)}{\left(1 + \frac{1}{2}z^{-1}\right)\left(1 - \frac{1}{3}z^{-1}\right)}$$

(2 pts)

$$Y(z) = \frac{A_1}{\left(1 + \frac{1}{2}z^{-1}\right)} + \frac{A_2}{\left(1 - \frac{1}{3}z^{-1}\right)}$$

(1 pt)

$$\frac{\left(1-\frac{1}{2}z^{-1}\right)}{\left(1+\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{3}z^{-1}\right)} = \frac{A_1}{\left(1+\frac{1}{2}z^{-1}\right)} + \frac{A_2}{\left(1-\frac{1}{3}z^{-1}\right)}$$

Multiply bot side by $\left(1+\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{3}z^{-1}\right)$ (2 pts)

both sides

$$\left(1-\frac{1}{2}z^{-1}\right) = A_1\left(1-\frac{1}{3}z^{-1}\right) + A_2\left(1+\frac{1}{2}z^{-1}\right)$$

$$\Rightarrow \begin{cases} 1 = A_1 + A_2 \\ -\frac{1}{2} = -\frac{1}{3}A_1 + \frac{1}{2}A_2 \end{cases} \Rightarrow \begin{cases} A_1 = \frac{6}{5} \\ A_2 = -\frac{1}{5} \end{cases}$$

(3 pts: two equations, 2 pts: answers)

Therefore,

Terefore

$$\mathrm{Y(z)} = \frac{\frac{6}{5}}{\left(1+\frac{1}{2}z^{-1}\right)} + \frac{-\frac{1}{5}}{\left(1-\frac{1}{3}z^{-1}\right)}$$

c. There are three possible ROCs for this Y(z).

Let $\mathrm{Y_1(z)} = \frac{\frac{6}{5}}{\left(1+\frac{1}{2}z^{-1}\right)} \overset{ZT}{\leftrightarrow} y_1[n]$, $\mathrm{Y_2(z)} = \frac{-\frac{1}{5}}{\left(1-\frac{1}{3}z^{-1}\right)} \overset{ZT}{\leftrightarrow} y_2[n]$

ROC{Y(z)} = ROC{H(z)} intersection ROC{X(z)}

or ROC{Y(z)} = ROC{Y_1(z)} intersection ROC{Y_2(z)}

The statement that the system is causal means that ROC{H(z)} = {z: |z| > 1/2}, since h[n] = ZT^{-1}(H(z)) must be right-sided. So ROC{Y(z)} = {z: |z| > 1/2} is the only possibility in this case, i.e. nothing can be left-sided. So please remove cases 2) and 3).

1) ROC{ Y(z) } = { z: $|z| > \frac{1}{2}$ }, then $y_1[n]$ and $y_2[n]$ are both right-sided

Use ZT pair

$$\mathrm{a^n u[n]} \overset{\mathrm{ZT}}{\leftrightarrow} \frac{1}{1-\mathrm{a}z^{-1}}, |z| > |a|$$

$$\mathrm{y[n]} = \frac{6}{5}\left(-\frac{1}{2}\right)^{\mathrm{n}}\mathrm{u[n]} - \frac{1}{5}\left(\frac{1}{3}\right)^{\mathrm{n}}\mathrm{u[n]}$$

2) ROC{ Y(z) } = { z: $\frac{1}{3} < |z| < \frac{1}{2}$ }, then $y_1[n]$ is left-sided and $y_2[n]$ is right-sided.

For $y_1[n]$, use

$$-\mathrm{a^n u[-n-1]} \overset{\mathrm{ZT}}{\leftrightarrow} \frac{1}{1-\mathrm{a}z^{-1}}, |z| < |a|$$

$$y_1[n] = \frac{6}{5}(-1)\left(-\frac{1}{2}\right)^n u[-n-1]$$

$$y[n] = -\frac{6}{5}\left(-\frac{1}{2}\right)^n u[-n-1] - \frac{1}{5}\left(\frac{1}{3}\right)^n u[n]$$

3) ROC{ Y(z) } = { z: $|z| < \frac{1}{3}$ }, then $y_1[n]$ and $y_2[n]$ are both left sided.

use

$$-a^n u[-n-1] \overset{ZT}{\leftrightarrow} \frac{1}{1-az^{-1}}, |z| < |a|$$

$$y_2[n] = -\frac{1}{5}(-1)\left(\frac{1}{3}\right)^n u[-n-1]$$

$$y[n] = -\frac{6}{5}\left(-\frac{1}{2}\right)^n u[-n-1] + \frac{1}{5}\left(\frac{1}{3}\right)^n u[-n-1]$$

(2 pts: three ROCs)

(4 x 2pts: 4 ZT results)

ECE 438 - Exam 2 - Solution

2. (25 pts) Fast Fourier Transform Algorithm

   a. (2) Calculate the *approximate* number of complex operations (COs) required to compute a 9-point DFT by directly evaluating the 9-point DFT sum. Here a complex operation is taken to mean 1 complex addition and 1 complex multiplication.

   b. (8) Derive a complete set of equations to show how a 9-point Discrete Fourier Transform (DFT) can be calculated in two stages via decimation-in-time.

   c. (8) Draw complete flow diagram for your 9-point DFT

   d. (3) Based on your answer to part (c) above, list the complete ordering of the 9-point input to your 9-point FFT algorithm.

   e. (4) Based on your answer to part (c) above, calculate the *approximate* number of complex operations (COs) required to compute a 9-point DFT using your FFT algorithm.

Solution:-

(a) Formula for DFT,

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi kn}{N}}$$

So, for each $X[k]$, there are N multiplications & $N-1$ additions needed.

$\therefore$ Approximately $O(N^2)$ COs are required.

So, No. of COs $= N^2 = 9^2 = 81$

(b) 9pt. DFT = 3 × 3 pt. DFTs.

So, $X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi kn}{N}}$

$$X^{(9)}[k] = \sum_{n=0}^{8} x[n]\, e^{-j\frac{2\pi kn}{9}}$$

Now splitting into three 3-pt DFTs.

$$X^{(9)}[k] = \sum_{n=0}^{8} x[n]\, e^{-j\frac{2\pi kn}{9}}$$

let $n = 3m+p$.

where $m = 0, \dots, 2$

$p = 0, \dots, 2.$

Define $x_p[m] = x[3m+p]$, $m = 0, 1, 2$

$p = 0, 1, 2.$

$$X^{(9)}[k] = \sum_{p=0}^{2} \sum_{m=0}^{2} x_p[m]\, e^{-j\frac{2\pi k(3m+p)}{9}}$$

$$= \sum_{p=0}^{2} \sum_{m=0}^{2} x_p[m]\, e^{-j\frac{2\pi km}{3}} \cdot e^{-j\frac{2\pi kp}{9}}$$

* Note a 3-pt DFT can be written as.

$$X_p^{(3)}[k] = \sum_{m=0}^{3} x_p[m]\, e^{-j\frac{2\pi km}{3}}.$$

So, using this fact in above we get,

$$X^{(9)}[k] = \sum_{p=0}^{2} X_p^{(3)}[k]\, e^{-j\frac{2\pi kp}{9}}$$

$$= \sum_{p=0}^{2} X_p^{(3)}[k]\, W_9^{kp}$$

where $W_9^{kp} = e^{-j\frac{2\pi kp}{9}}$

for $k = 0, \dots, 8$

$\&\ p = 0, 1, 2$

$p=0$

$k=0$

Also,

$$X_p^{(3)}[k] = \sum_{m=0}^{2} x[3m+p]\, e^{-j\frac{2\pi km}{3}}$$

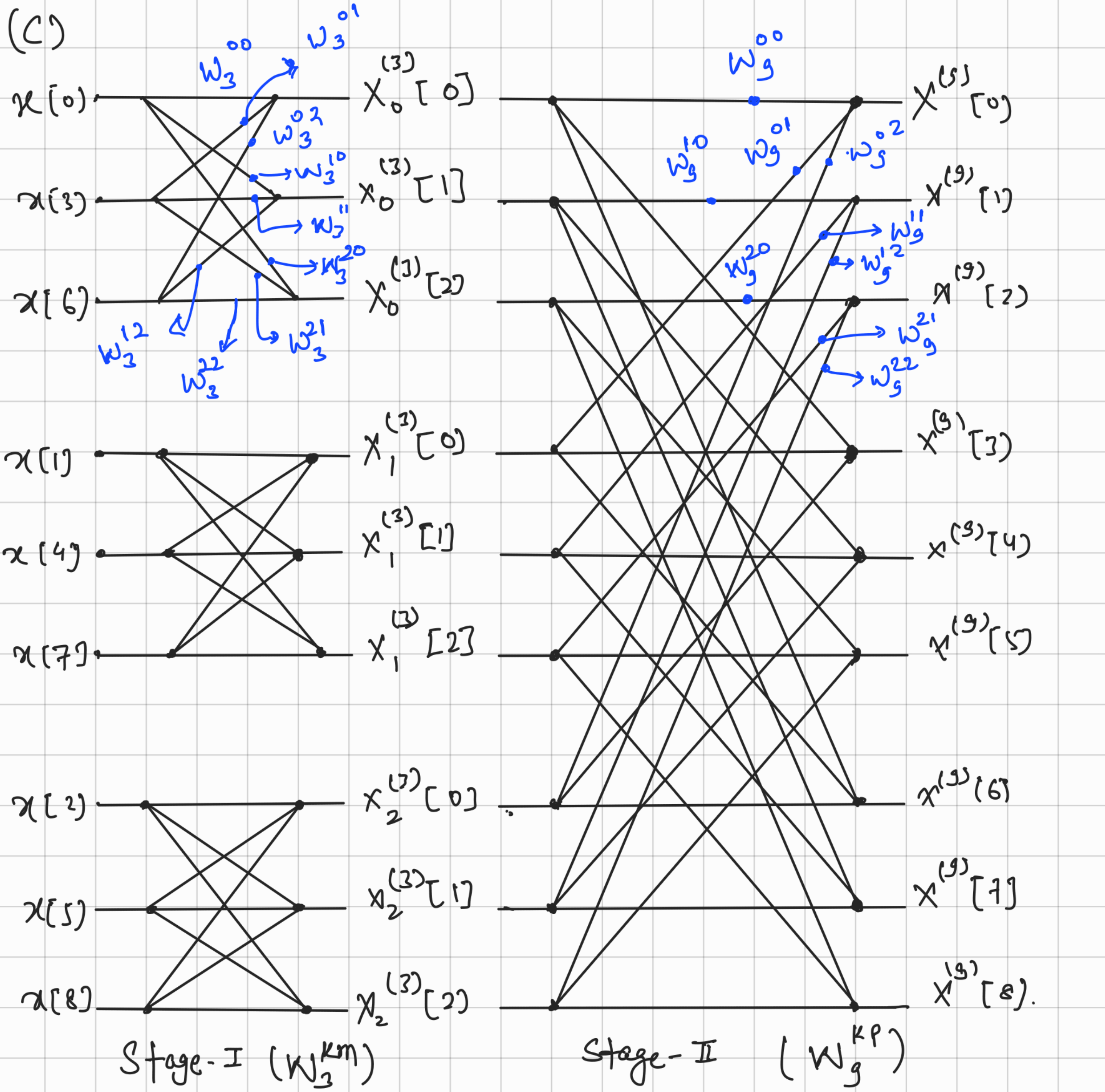
$$= \sum_{m=0}^{2} x[3m+p]\, W_3^{km}$$

where $W_3^{km} = e^{-j\frac{2\pi km}{3}}$

for $k=0,1,2$

& $m=0,1,2$.

(C)



(d) Complete ordering

p	m	3m+p	x[3m+p]
0	0	0	x[0]
0	1	3	x[3]
0	2	6	x[6]
1	0	1	x[1]
1	1	4	x[4]
1	2	7	x[7]
2	0	2	x[2]
2	1	5	x[5]
2	2	8	x[8]

(e) For Stage I.

$k = 0, 1, 2$

Each point in a 3-pt DFT requires → 2 COs

So, total no. of COs in stage 1

$= 2 \times 3 \times 3 = 18$ COs.

For Stage – II.

$K = 0, 1, \ldots\ 8$

Each point (value of K) requires 2 COs.

So in total 9 points (values of K) require → $2 \times 9 = 18$ COs.

∴ Total number of COs = (18 + 18) COs
= 36 (CO)s

3. (25 pts) Spectral analysis via the DFT

Consider the 32-point signal $x[n]=\cos\left(\frac{2\pi(3)}{32}n\right), n=0,...,31$.

a. (6) Determine an exact expression for the 32-point discrete Fourier transform (DFT) $X^{(32)}\left[k\right], k=0,...,31$ of this signal.

b. (2) Determine the approximate values of k where you would find the peaks in the DFT.

c. (1) Are there any leakage or picket fence effects in this case? Why or why not?

Now consider the 32-point signal $x[n]=\cos\left(\frac{2\pi(7)}{64}n\right), n=0,...,31$.

d. (11) Determine an exact expression for the 32-point discrete Fourier transform (DFT) $X^{(32)}\left[k\right], k=0,...,31$ of this signal in terms of the function

$\text{psinc}_N(\omega)=\frac{\sin(\omega N/2)}{\sin(\omega/2)}$. Be sure to specify the value for N.

e. (4) Determine the approximate values of k where you would find the peaks in the DFT.

f. (1) Are there any leakage or picket fence effects in this case? Why or why not?

Exam-2

Ans: 3

(a) $x[n] = \cos\left(\frac{2\pi(3)}{32} n\right)$, $n = 0, \ldots, 31$

$X[k] = \text{DTFT}\{x[n]w[n]\}$ $w = \frac{2\pi k}{N}$, $N = 32$

$w[n] = \begin{cases} 1, & 0 \le n \le N-1 \\ 0, & \text{else} \end{cases}$

$X(w) = \pi\left[\delta\left(w - \frac{2\pi(3)}{32}\right) + \delta\left(w + \frac{2\pi(3)}{32}\right)\right]$, $|w| \le \pi$

$W(w) = e^{-jw\frac{(32-1)}{2}} \frac{\sin\left(\frac{32w}{2}\right)}{\sin(w/2)}$

$\therefore X[k] = \frac{1}{2\pi} X(w) * W(w) = \frac{1}{2}\left[W\left(w - \frac{2\pi(3)}{32}\right) + W\left(w + \frac{2\pi(3)}{32}\right)\right]$, $|w| \le \pi$

$w = \frac{2\pi k}{32}$

$\therefore X[k] = \frac{1}{2}\left[W\left(\frac{2\pi k}{32} - \frac{2\pi(3)}{32}\right) + W\left(\frac{2\pi k}{32} + \frac{2\pi(3)}{32}\right)\right]$

$= \frac{1}{2}\left[W\left(\frac{2\pi}{32}(k-3)\right) + W\left(\frac{2\pi}{32}(k+3)\right)\right]$

Non-zero only for $k = 3$

Non-zero only for $k = -3 + 32 = 29$

$$\therefore X[k] = \frac{1}{2}\left[32\,\delta[k-3] + 32\,\delta[k-29]\right]$$

$$= 16\,\delta[k-3] + 16\,\delta[k-29]$$

(b) $\frac{2\pi k}{N} = \frac{2\pi(3)}{32}$, $N = 32$

$\Rightarrow k = 3$ and $k = 29$

(c) There are no leakage effects because the frequency of the sinosoid belongs to the set of analysis frequencies used to compute the DFT.

(d) $x[n] = \cos\left(\frac{2\pi(7)}{64} n\right)$, $n = 0, \dots 31$

$$X[k] = \frac{1}{2}\left[W\left(2\pi\frac{k}{32} - \frac{2\pi(7)}{64}\right) + W\left(2\pi\frac{k}{32} + \frac{2\pi(7)}{64}\right)\right]$$

$$= \frac{1}{2}\left[W\left(\frac{2\pi}{32}\left(k - \frac{7}{2}\right)\right) + W\left(\frac{2\pi}{32}\left(k + \frac{7}{2}\right)\right)\right]$$

$$= \frac{1}{2}\, e^{-j\omega(31/2)}\left[\operatorname{psinc}_{32}\left(\frac{2\pi}{32}\left(k - \frac{7}{2}\right)\right) + \operatorname{psinc}_{32}\left(\frac{2\pi}{32}\left(k + \frac{7}{2}\right)\right)\right]$$

(e) Peaks at :

$k = \frac{7}{2}$ and $k = -\frac{7}{2} + 32 = \frac{57}{2}$

$= 3.5$ $\qquad\qquad = 28.5$

(f) We will see leakage effects as k is ~~a~~ ~~decimal~~ not an integer which means the frequency of the sinosoid does not belong to the set of analysis frequencies used to compute the DFT.

4. (25 pts) Consider a random variable X with density function

$$f_X(x) = \begin{cases} 1.5x^2, & -1 \le x \le 1 \\ 0, & \text{else} \end{cases} .$$

a. (4) Carefully sketch *by hand* the density function $f_X(x)$. Be sure to dimension both axes.

b. (9) Find the mean and variance of X.

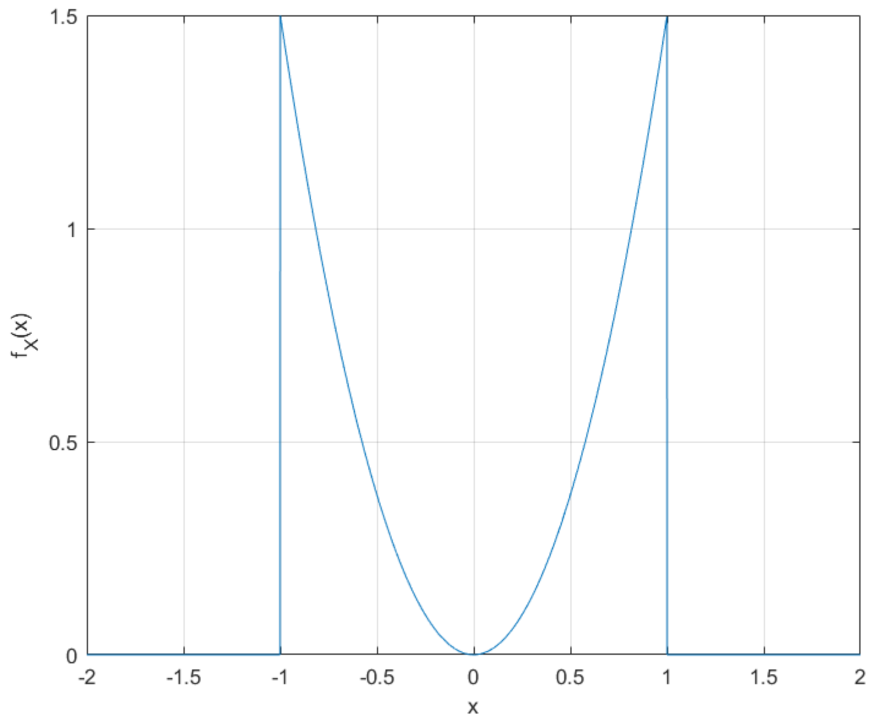
c. (8) Suppose we generate a new random variable $Y = Q(X)$ by quantizing X according to the following 3-level *uniform* quantizer:

$$Q(x) = \begin{cases} -\frac{2}{3}, & -1 \le x < -\frac{1}{3} \\ 0 & -\frac{1}{3} \le x < \frac{1}{3} \\ \frac{2}{3}, & \frac{1}{3} \le x < 1 \end{cases} .$$

Determine the *approximate* mean-squared quantization error $\varepsilon = E\left\{|Y - X|^2\right\}$ using the expression $\varepsilon_{\text{approx}} = \dfrac{\Delta^2}{12}$

d. (4) Use your answers from parts (b) and (c) to calculate the approximate signal-to-noise ratio due to quantization.

(a) The plot is attached below.
(2 pt) Correct plot. (1 pt) Horizontal axis. (1 pt) Vertical axis.



(b)

$$\text{(3 pts) } \mu_X = \mathbf{E}[X] = \int_{-1}^{1} x \cdot (1.5x^2)dx = \frac{1.5}{4}x^4|_{-1}^{1} = 0$$

$$\text{(3 pts) } \mathbf{E}[X^2] = \int_{-1}^{1} x^2 \cdot (1.5x^2)dx = \frac{1.5}{5}x^5|_{-1}^{1} = \frac{3}{5}$$

$$\text{(3 pts) } \sigma_x^2[X] = \mathbf{E}[X^2] - \mathbf{E}[X]^2 = \frac{3}{5} - (0)^2 = \frac{3}{5} = 0.6$$

(c) For the given 3-level quantizer:

$$\text{(4 pts) } \Delta = \frac{Q_{max} - Q_{min}}{\text{No. of Levels} - 1} = \frac{\frac{2}{3} - (-\frac{2}{3})}{3 - 1} = \frac{2}{3}$$

$$\text{(4 pts) } \epsilon_{approx} = \frac{\Delta^2}{12} = \frac{1}{18}$$

1/27

I did not ask for this; so we should not include it. It will cause confusion.

The exact case results at $(\frac{3x^5}{10} + \frac{x^4}{2} + \frac{2x^3}{9})|_{-1}^{-1/3} + \frac{3x^5}{10}|_{-1/3}^{1/3} + (\frac{3x^5}{10} - \frac{x^4}{2} + \frac{2x^3}{9})|_{1/3}^{1} = 0.4033$

(d)

$$\text{(4 pts) } SNR = 10\log_{10}\frac{\mathbf{E}[X^2]}{\epsilon_{approx}} = 10\log_{10}\frac{54}{5} \approx 10.3342 \text{ dB}$$

(3/5)*27