15 February 2023 **Name:____________________**

ECE 438 **Exam No. 1** **Spring 2023**

- You have 50 minutes to work the following four problems.
- Be sure to show all your work to obtain full credit.
- The exam is closed book and closed notes. Smart watches and mobile phones must be put away.
- Calculators are **not** permitted.

1. (25 pts.) Consider the linear, time-invariant system defined by the difference equation

$$y[n]=\frac{1}{2}\left\{x[n]+x[n-3]\right\}.$$

a. (5) Find the response to the input $x[n]=\begin{cases} 1, & -1\le n\le 1 \\ 0, & \text{else} \end{cases}$

b. (10) Find a simple expression for the frequency response $H(\omega)$ of this system.

c. (4) Find a simple expression for the magnitude $|H(\omega)|$ of the frequency response.

d. (3) Find a simple expression for the phase $\angle H(\omega)$ of the frequency response.

e. (3) Carefully sketch $|H(\omega)|$ and $\angle H(\omega)$. Be sure to dimension all important quantities on both the horizontal and vertical axes.

1. (25 pts.) Consider the linear, time-invariant system defined by the difference equation

$$y[n] = \frac{1}{2}\{x[n] + x[n-3]\}$$

a. (5) Find the response to the input $x[n] = \begin{cases} 1, & -1 \le n \le 1 \\ 0, & else \end{cases}$

b. (10) Find a simple expression for the frequency response $H(\omega)$ of this system.

c. (4) Find a simple expression for the magnitude $|H(\omega)|$ of the frequency response.

d. (3) Find a simple expression for the phase $\underline{/H(\omega)}$ of the frequency response.

e. (3) Carefully sketch $|H(\omega)|$ and $\underline{/H(\omega)}$. Be sure to dimension all important quantitities on both the horizontal and vertical axes.

Solution:

a.

$$-1 \le n-3 \le 1 \implies 2 \le n \le 4 \qquad \text{(2 pts)}$$

$$y[n] = \frac{1}{2}\{x[n] + x[n-3]\}$$

$$= \begin{cases} \frac{1}{2}, & -1 \le n \le 1,\ 2 \le n \le 4 \\ 0, & else \end{cases} \qquad \text{(2 pts)}$$

$$= \begin{cases} \frac{1}{2}, & -1 \le n \le 4 \\ 0, & else \end{cases} \qquad \text{(1 pt)}$$

b. Consider $x[n] = e^{j\omega n}$, then (1 pt)

$$y[n] = \frac{1}{2}\left(e^{j\omega n} + e^{j\omega(n-3)}\right)$$

$$= \frac{1}{2}e^{j\omega n}\left(1 + e^{-j3\omega}\right)$$

$$= \frac{1}{2}e^{j\omega n}e^{-j\frac{3}{2}\omega}\left(e^{j\frac{3}{2}\omega} + e^{-j\frac{3}{2}\omega}\right)$$

$$= e^{j\omega n}e^{-j\frac{3}{2}\omega}cos\left(\frac{3}{2}\omega\right) \qquad \text{(using trick: 3 pts)}$$

(correct derivation: 2 pts)

For a LTI system with complex exponential input:

$$y[n] = H(\omega)x[n] \qquad \text{(3 pts)}$$

$$\therefore H(\omega) = e^{-j\frac{3}{2}\omega}cos\left(\frac{3}{2}\omega\right) \qquad \text{(1 pt)}$$

c.

$$|H(\omega)| = \left|e^{-j\frac{3}{2}\omega}\right|\left|cos\left(\frac{3}{2}\omega\right)\right| \qquad \text{(1 pt)}$$

$$|H(\omega)| = \left|cos\left(\frac{3}{2}\omega\right)\right| \qquad \text{(1 pt)}$$

(magnitude of the first term = 1: 2pts)

d.

$$\angle H(\omega) = \angle e^{-\frac{j3}{2}\omega} + \angle \cos\left(\frac{3}{2}\omega\right)$$

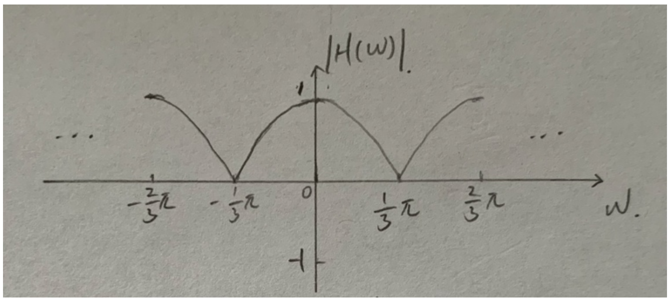
$$\angle e^{-\frac{j3}{2}\omega} = -\frac{3}{2}\omega \qquad \text{(1 pt)}$$

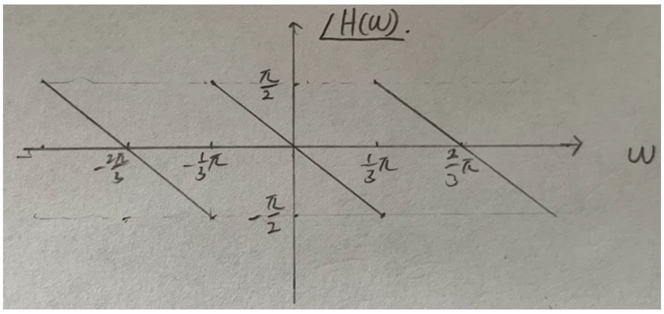
$$\angle \cos\left(\frac{3}{2}\omega\right) = \begin{cases} 0, & \cos\left(\frac{3\omega}{2}\right) > 0 \\ \pm\pi, & \cos\left(\frac{3\omega}{2}\right) < 0 \end{cases} \qquad \text{(1 pt)}$$

So counting everything, we have

$$\angle H(\omega) = \begin{cases} -\frac{3}{2}\omega, & \cos\left(\frac{3\omega}{2}\right) > 0 \\ -\frac{3}{2}\omega \pm \pi, & \cos\left(\frac{3\omega}{2}\right) < 0 \end{cases} \qquad \text{(1 pt)}$$

e.





(correct period: 1pt)

(correct magnitude (vertical axis): 1pt)

(correct shape: 1pt)

2. (25 pts.) Consider a linear, time-invariant system with unit sample (impulse) response

$$h[n] = \left(\frac{1}{2}\right)^n u[n].$$

a. (20) Find the response of this system $y[n]$ to the input

$$x[n] = \begin{cases} 1, & 0 \le n \le 9 \\ 0, & \text{else} \end{cases},$$

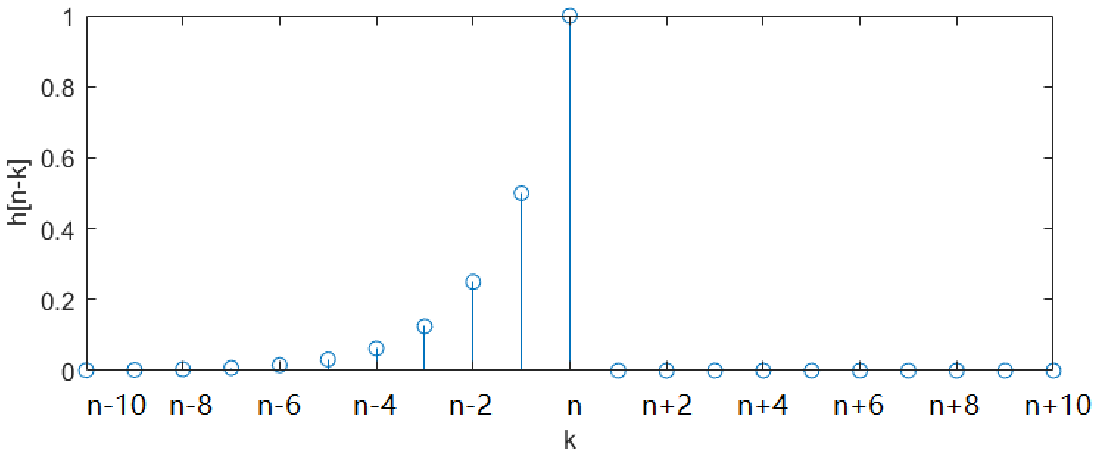
by evaluating the convolution $y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$.

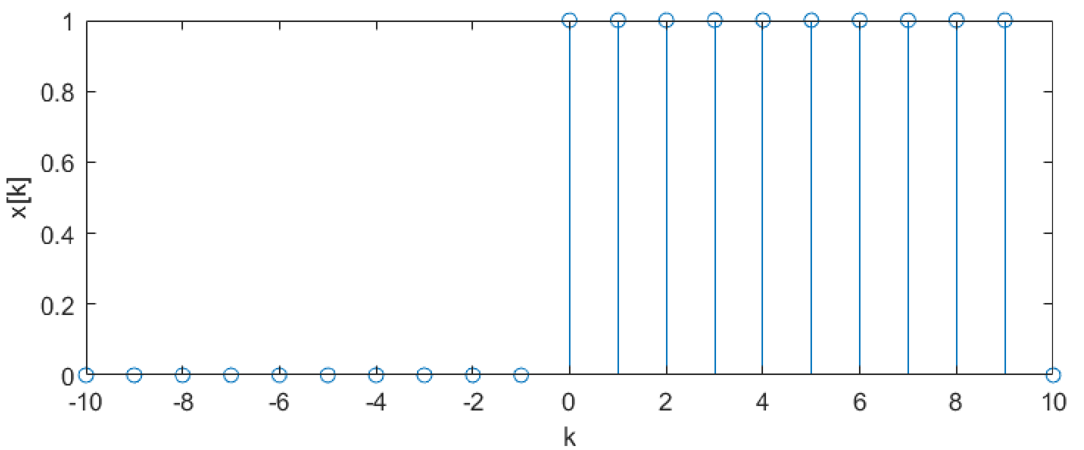
Your solution for $y[n]$ should be an analytical expression or expression(s) for the signal. It should not contain any summation signs $\sum$.

b. (5) Carefully sketch the function $y[n]$.

1.

2. (a) The plots of $h[n-k]$ and $x[k]$ are shown below:





$$
\begin{aligned}
y[n] &= \sum_{k=-\infty}^{\infty} x[k]h[n-k] \\
&= \sum_{k=0}^{9} h[n-k] \\
&= \sum_{k=0}^{9} (\frac{1}{2})^{n-k} u[n-k]
\end{aligned}
$$

Case 1: $n < 0$ $y[n] = 0$
Case 2: $n \geq 0$ **and** $n < 9 \rightarrow 0 \leq n < 9$

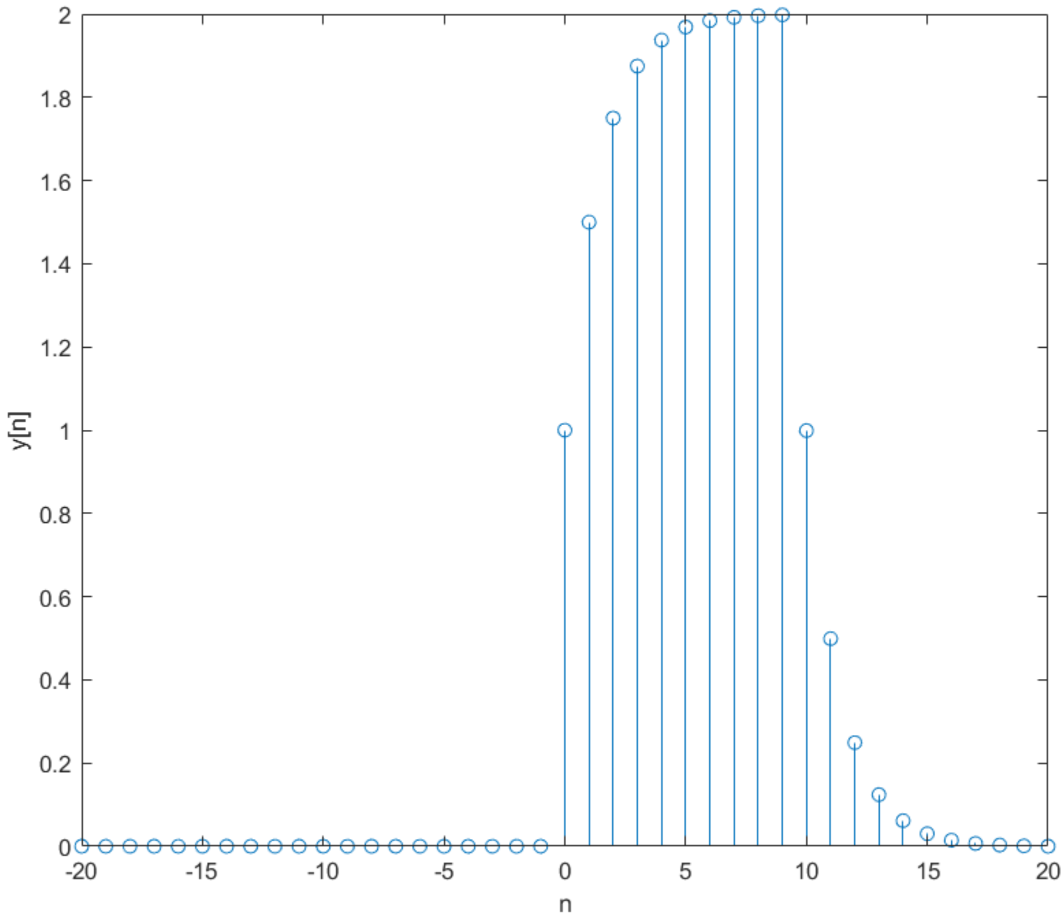
$$\begin{aligned} y[n] &= \sum_{k=0}^{n} (\frac{1}{2})^{n-k} \\ &= \sum_{k=0}^{n} 2^{k-n} \\ &= \frac{2^{-n}(1 - 2^{n+1})}{1-2} \\ &= 2 - 2^{-n} \\ &= 2 - (\frac{1}{2})^{n} \end{aligned}$$

Case 3: $n \geq 9$

$$\begin{aligned} y[n] &= \sum_{k=0}^{9} (\frac{1}{2})^{n-k} \\ &= \sum_{k=0}^{9} 2^{k-n} \\ &= \frac{2^{-n}(1 - 2^{10})}{1-2} \\ &= (\frac{1}{2})^{n-10} - (\frac{1}{2})^{n} \end{aligned}$$

= (1/2)^n *(2^10 - 1)
= 1023*(1/2)^n

(b) The plot is shown below:



3. (25) Consider the real-valued continuous-time signal $x(t)$ defined by

$$x(t) = \cos(2\pi t/6)$$

a. (5) Find a simple expression for the CTFT $X(f)$ of $x(t)$

Now, let the continuous-time signal $y(t)$ be defined as

$$y(t) = \begin{cases} x(t), & |t| < 3 \\ 0, & \text{else} \end{cases},$$

where $x(t)$ is defined as above.

b. (15) Find a simple expression for the CTFT $Y(f)$ of $y(t)$. Your answer should not include any operators, such as convolution, rep, or comb.

c. (5) Sketch $Y(f)$. Be sure to dimension all important quantities on both the horizontal and vertical axes

Q.3. Solution:

Given,

$$x(t) = \cos(2\pi t/6).$$

(a) Find simple expression of $X(f)$ of $x(t)$.

→ $x(t) = \cos(2\pi t/6)$

$$= \frac{e^{j2\pi t/6} + e^{-j2\pi t/6}}{2}$$

So, $X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt$

$$= \frac{1}{2}\left[\int_{-\infty}^{\infty} e^{j2\pi t/6}\, e^{-j2\pi f t}\, dt + \int_{-\infty}^{\infty} e^{-j2\pi t/6}\, e^{-j2\pi f t}\, dt\right]$$

$$= \frac{1}{2}\left[\mathcal{F}\{e^{j2\pi t/6}\} + \mathcal{F}\{e^{-j2\pi t/6}\}\right]$$

$$X(f) = \frac{1}{2}\left[\delta(f - 1/6) + \delta(f + 1/6)\right]$$

(b)

Given,

$$y(t) = \begin{cases} \cos(2\pi t/6), & |t| < 3 \\ 0, & \text{else} \end{cases}$$

$Y(f) = ?$

→

$$y(t) = \text{rect}\left(\frac{t}{6}\right) \times \cos(2\pi t/6)$$

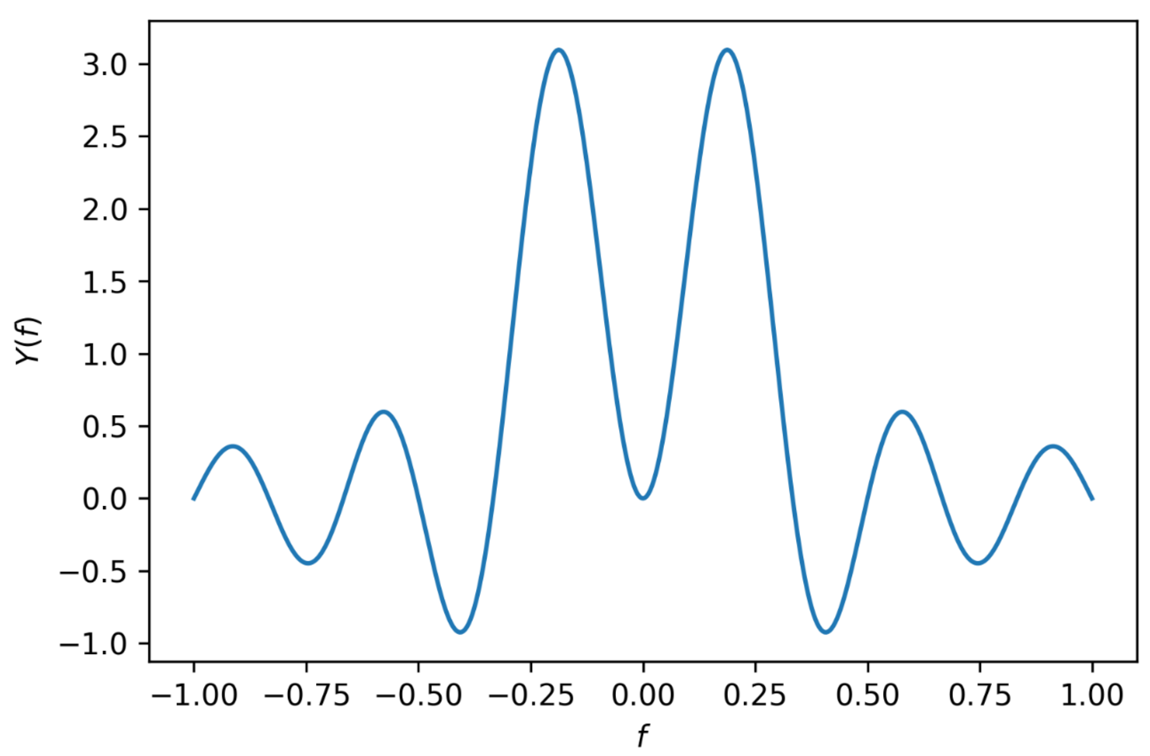
So,

$$Y(f) = 6\,\text{sinc}(6f) * \frac{1}{2}\left[\delta\left(f-\frac{1}{6}\right) + \delta\left(f+\frac{1}{6}\right)\right]$$

$$= 3\,\text{sinc}(6f) * \left[\delta\left(f-\frac{1}{6}\right) + \delta\left(f+\frac{1}{6}\right)\right]$$

$$= 3\,\text{sinc}\left(6\left(f-\frac{1}{6}\right)\right) + 3\,\text{sinc}\left(6\left(f+\frac{1}{6}\right)\right)$$

(c)



4. (25 pts) The signal $x(t) = \cos(2\pi(2)t)$ is sampled at interval $T = 1/3$ sec to yield $x_s(t)$

a. (6 pts) Use the comb operator to represent the sampled signal $x_s(t)$ as a train of impulses. Provide an expression for $x_s(t)$ that consists of a train of impulses.

b. (6 pts) Find an expression for the CTFT $X_s(f)$ of $x_s(t)$. Your answer should not contain any operators, such as rep or comb.

c. (6 pts) Sketch $X_s(f)$. Be sure to dimension all important quantities on both the horizontal and vertical axes in your plot.

d. (7 pts) The signal $x_s(t)$ is input to an ideal low-pass filter with unity gain in the passband, and cutoff at 1.5 Hz. Find a simple time-domain expression for the filter output $x_r(t)$.

Ans: 4

(a) $x(t) = \cos[2\pi(2)t]$, $F = 2\,Hz$

$T = \frac{1}{3}\,s$

$\Rightarrow F_s = 3\,Hz$ $\therefore F_s < 2F$ [Aliasing will be present]

$x_s(t) = comb_{1/3}[x(t)]$

$= comb_{1/3}[\cos[2\pi(2)t]]$

$\Rightarrow x_s(t) = \sum_{k=-\infty}^{\infty} \cos\left(\frac{4\pi}{3}k\right)\delta\left(t - \frac{4\pi}{3}k\right)$

k/3

(b) $x(t) = \cos[2\pi(2)t]$

$X(f) = \frac{1}{2}[\delta(f-2) + \delta(f+2)]$

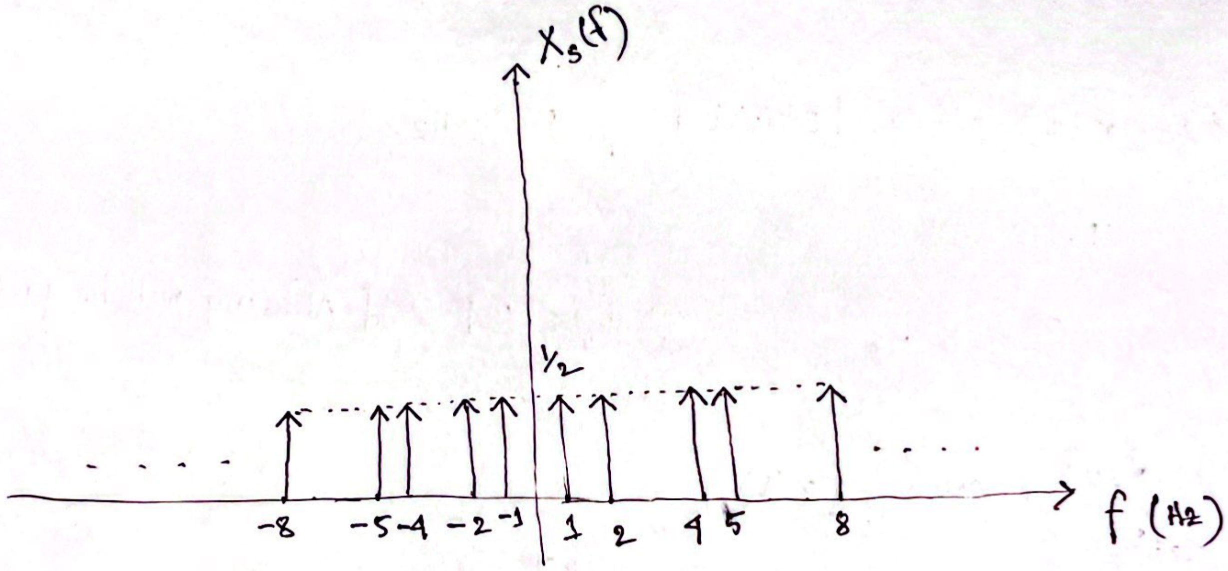
$x_s(t) = comb_{1/3}[x(t)]$

$X_s(f) = 3\,rep_3[X(f)]$

$= \sum_{k=-\infty}^{\infty} X(f-3k)$

$\therefore X_s(f) = \frac{1}{2}\sum_{k=-\infty}^{\infty}[\delta(f-2-3k) + \delta(f+2-3k))$

(c)



(d)

$x_s(t) \longrightarrow$ [LPF] $\longrightarrow x_r(t)$

$\omega_c = 1.5$ Hz

$G = 1$

$$X_r(f) = \frac{1}{2}\left[\delta(f-1) + \delta(f+1)\right]$$

$$\therefore x_r(t) = \cos\left[2\pi(1)t\right]$$

$$= \cos(2\pi t)$$