Evolution of Drop Size Distributions in Fully Developed Turbulent Pipe Flow of a Liquid–Liquid Dispersion by Breakage

Nandkishor K. Nere and Doraiswami Ramkrishna*

School of Chemical Engineering, Purdue University, West Lafayette, Indiana 47907

The population balance equation for the evolution of drop size distributions in fully developed turbulent flow of a liquid–liquid dispersion in a circular pipe has been solved “exactly” using spectral expansion of the self-adjoint diffusion operator with a radially varying diffusion coefficient to obtain the number density at any location in the pipe. The breakage frequency is allowed to vary with position, although the size distribution of broken fragments is assumed to satisfy a form of similarity assumed in the work of Narsimhan et al. (AIChE J. 1980, 26, 991; 1984, 30, 457) that rids it of explicit spatial dependence. Of course, insofar as numerical methods are used to calculate the spectral data (eigenvalues and eigenvectors), such an exact solution is still to be regarded as approximate. Furthermore, because the solution is expressed in terms of a transient well-mixed batch dispersion evolving by breakage, the actual number density may be obtained by any of the methods for solving population balance equations in this simpler setting. In this paper, we use the method of Kumar and Ramkrishna (Chem. Eng. Sci. 1996, 51 (8), 1311) to solve the population balance equation for a batch system. This discretization method is also incorporated into a detailed simulation of the population balance equation in combination with computational fluid dynamics using the control volume approach. The latter method is incomparably demanding with regard to memory and computational time and consequently irrelevant. Self-similar solutions are obtained by including spatial scaling from the spectral expansion and particle size scaling from the work of Sathyagal et al. (Comput. Chem. Eng. 1995, 19, 437).

1. Introduction

We are pleased to contribute to a volume felicitating Professor Darsh Wasan, noted for his contributions to colloid and interfacial science. This paper is concerned with dispersion by droplet breakage due to turbulent flow in a pipe. However, it is also of consequence to particulate processes in general, in which particle breakage plays a significant role.

The process of breakage of particulate matter is important in numerous unit operations encountered in process chemical and allied industries. Examples include the transport of gas–liquid and liquid–liquid dispersions, particulate matter through conveyers accompanied by comminution, and so on. Breakage occurs normally in turbulent flow environments, where the local turbulent force responsible for particle breakup varies spatially and temporally. The steady-state flow pattern is complex with spatial diversity in various flow properties, leading to spatially dependent breakage phenomena that make the solution of the population balance equation (PBE) difficult. Recent research in this area has triggered the integration of computational fluid dynamics (CFD) into various methods for the solution of PBEs for well-mixed systems.

Our objective in this paper is to present rather simple solutions to PBEs for a liquid–liquid dispersion evolving purely by breakage in fully developed turbulent flow. The drops are assumed to be in the size range not requiring consideration of inertia. Breakup is assumed to result from turbulent forces so that the breakage frequency is assumed to be a function of the local energy dissipation. The energy dissipation field arises from the $k$–$\epsilon$ model for the flow. In view of the no-inertia assumption, the equation of motion is uncoupled from the population balance. The population balance is, however, coupled to the fluid momentum equation through the appearance in the former of the energy dissipation and the position-dependent turbulent diffusion coefficient for the particles. The number density is assumed to be small enough for coalescence between droplets to be negligible. Because the breakage model provides for self-similar behavior in batch dispersions as shown by Sathyagal et al., it is also of interest to see how self-similarity might manifest in turbulent flows of this kind.

The paper has been organized as follows. We briefly scope the literature on prior work on droplet breakage in turbulent flows. Next, we present the population balance in terms of the cumulative volume fraction by virtue of its simplicity, although the analysis is applicable just as well to the population balance in its usual form. A transformation is introduced by extracting the radial dependence of the breakage frequency into the axial derivative in the form of transformed time. Next we provide a self-adjoint linear operator formalism (for the radial eddy diffusion of the particles) that leads to an instant solution in terms of the spectral representation of the operator. Axial diffusion is neglected in the analysis, implying a suitably large axial Peclet number. Because the radial diffusion coefficient also depends on the energy dissipation, the spectral representation has to be obtained numerically. This, however, is accomplished using Matlab without either extensive programming or computation times. We present the
solution in terms of the cumulative volume fraction or the number density function for comparison with detailed numerical solutions using CFD.

2. Drop Breakage in Turbulent Flow: Prior Work

The problem of turbulent breakage in turbulent flow systems has been investigated by many researchers. Early work (Kolmogorov, Levich, and Hinze) focused on estimation of the maximum stable diameter of a drop in turbulent flow. Collins and Knudsen have presented measurements of drop size distributions with photographic evidence of deformed near-breakage drops. High deformation and breakup appeared to be much more frequent close to the wall, in the buffer layer, than in the turbulent core in conformity with Sleicher’s observation. Karabelas’ investigation focused on steady-state drop size distribution of an aqueous phase dispersed in an organic continuous phase. However, because this work was not concerned with the detail of drop size distributions, the issue of a breakage frequency was outside its domain. The first considerations toward obtaining a breakage frequency as a phenomenological input to PBEs were due to Coulaloglou and Tavlarides and Narsimhan et al., although they were based on somewhat different arguments. The energy dissipation rate appears in their frequencies. Ramkrishna presents a treatment, based on a model by Lagisetty et al., designed for the maximum stable diameter, that obtains the breakage frequency.

Because drops in the turbulent core are exposed to lower dissipation rates and hence are less susceptible to breakup, while those near the wall are considerably more breakable, it should be transparent that the gradation in the energy dissipation rate in the flow should substantially influence drop breakup and should be a feature of the population balance model. Although the frequencies derived by Coulaloglou and Tavlarides and Narsimhan et al. do contain the energy dissipation rate, we rely on the work of Sathyagal et al., who have experimentally determined drop breakage rates by an inverse problem approach based on self-similarity. Thus, they have determined breakage frequencies, as well as the size distribution of daughter droplets, for different rotational speeds in a well-mixed stirred tank. The cumulative volume fraction of daughter droplets in the size range \((0, v)\) from breakage of a parent of volume \(v'\) is characterized by a single function of the ratio of the breakage frequency of daughter drops of volume \(v\) to that of the parent drop, which is a consequence of the assumption of similar breakage made first by Narsimhan et al. This breakage law implies that drops break favoring breakable fractions more and that larger drops will suffer more “thorough” breakup than smaller drops, which will tend to break into two equal-sized droplets. This is illustrated in Figure 1. The breakage functions so obtained by Sathyagal et al. predict through the PBE the transient drop volume densities quite accurately for different agitation speeds.

Because the energy dissipation rate can be calculated for the three different speeds at which the breakage functions became available from the work of Sathyagal et al., we are now in a position to fit a breakage frequency as a function of the local dissipation rate in the current work. This provides a reasonable starting point for our population balance model. On the other hand, the accuracy with which this model may satisfy the actual breakup process is not as much of a burning issue to this paper because its focus is on finding a remarkably simple solution to the determination of the local number density of drops from the detailed PBE.

The nonlinearity in the variation of the energy dissipation rate makes theoretical predictions based on average dissipation rates inaccurate. Nere and Ramkrishna have recently demonstrated this in aggregation processes in turbulent flow. Of particular interest to this paper is the fact that the similarity assumption for drop breakup rids the daughter drop size distribution of any spatial dependence. In what follows, we present a detailed population balance model based on the breakage functions of Sathyagal et al. fitted to account for the local energy dissipation rate.

3. PBE in Fully Developed Turbulent Pipe Flow

Consider steady, fully developed turbulent flow of lean liquid–liquid dispersion through a pipe. We assume that the drops are sufficiently small to neglect inertial effects. Thus, they have the same velocity as the fluid. Further, the effective density and kinematic viscosity of the dispersion can be assumed to be equal to that of the continuous phase because of the low volume fraction of the dispersed phase. The droplet collisions with the wall are assumed to be perfectly elastic and without breakage. Further, the turbulent pipe flow is inherently three-dimensional in nature, and hence the droplet number density is a function of spatial coordinates in addition to the internal coordinates (drop volume). Thus, the PBE in terms of the droplet number density can be written as

\[
\nabla U f_1(v, x) - \nabla [D(v, x) \nabla f_1(v, x)] = \int_0^\infty \Gamma(v', x) P(v, x|v') f_1(v', x) \, dv' - \Gamma(v, x) f_1(v, x)
\]

(1)

where \(U\) is the velocity vector and \(D(v, x)\) is the droplet diffusivity. The breakage functions include the breakage frequency \(\Gamma(v', x)\) for drops of volume \(v'\) and the mean number of droplets \(\nu(v', x)\) from the breakage of a droplet of volume \(v\) at location \(x\), and \(P(v|v', x)\) represents the daughter drop size distribution from the breakage at location \(x\) of a parent drop of volume \(v\). Assuming axisymmetry, eq 1, a fully developed flow, becomes in cylindrical coordinates.

![Figure 1. Turbulent breakage characteristics of drops for three different sizes as determined by the volume fraction density of broken fragments. Smaller drops yield fewer smaller fractions.](image-url)
When Sathyagal’s data are fitted, the foregoing break-in eq 2 is taken to be of the degenerate form:

\[
U_z(r) \frac{\partial f_1(v,r,z)}{\partial z} - \frac{1}{r} \frac{\partial}{\partial r} \left[ D(r) \frac{\partial f_1(v,r,z)}{\partial r} \right] = \int_0^\infty v' \psi(r) \Gamma(v',r) f_1(v',r,z) \, dv' 
\]

\[
(w') P(v,r|v') f_1(v',r,z) \, dv' - \Gamma(v,r) f_1(v,r,z) \quad (2)
\]

The diffusion coefficient is assumed to be independent of the particle size because it is assumed to be equal to the effective diffusion coefficient comprised of the eddy diffusion coefficient \((\mu/\rho\), which is given by the CFD simulation of the fully developed pipe flow using a low Reynolds number \(k-\epsilon\) model.

\[
D = \frac{\mu_t + \mu_M}{\rho} \quad (3)
\]

We have assumed in eq 3 the diffusion coefficient to be close to the kinematic viscosity, instead of the Brownian diffusion coefficient, very near the wall. This assumption is made to eliminate the size dependence introduced by the Brownian diffusion coefficient requiring a needlessly detailed calculation in a minuscule region near the wall. Furthermore, this is reasonable assumption in view of the Brownian diffusion coefficients for the noninertial droplet sizes on the order of \(10^{-10} \text{ m}^2/\text{s}\), which is considerably smaller than the local flow diffusivities. Thus, the assumption of zero inertia for the particles is extended all the way to the wall without seriously affecting the accuracy of the calculations. Assume that the particle feed occurs at some \(z = 0\) in the fully developed flow region in such a way that the particle size distribution is independent of the spatial distribution. Also we assume that the drops reflect off the wall, although other alternatives such as total or partial loss by deposition of the drops at the wall can be accommodated just as well. Thus, the boundary conditions to be satisfied by the function \(f_1(v,r,z)\) are as

\[
f_1(v,r,0) = f_{1.0}(v) \psi(r), \quad r = 0, R, \quad \frac{\partial f_1}{\partial r} = 0 \quad (4)
\]

The solution methodology in this paper is predicated on exploiting two special features assumed of the breakage functions. First, the breakage frequency \(\Gamma(v,r)\) in eq 2 is taken to be of the degenerate form:

\[
\Gamma(v,r) = \Gamma(v) \Gamma_g(r) \quad (5)
\]

When Sathyagal’s data are fitted, the foregoing breakage frequency has the following expressions:

\[
\Gamma(v) = \exp[-c_3 \ln v^2 + c_4 \ln v + c_5 v^{1.0}], \quad \Gamma_g(r) = c_1 e(r)^{c_2} \quad (6)
\]

where \(c_1 = 2.142 \times 10^4, c_2 = 2.11, c_3 = 0.011 543 3, c_4 = 0.266 947, \) and \(c_5 = 4.9628 \times 10^4.\) Before we shed light on the daughter drop size distribution \(P(v'|v)\), it is desirable to provide the form of eq 2 for the cumulative volume fraction \(F_1(v,r,z)\), defined by

\[
F_1(v,r,z) = \frac{1}{\Phi} \int_0^v f_1(v',r,z) \, dv', \quad \Phi = \int_0^\infty f_1(v',r,z) \, dv' \quad (7)
\]

to obtain

\[
U_z(r) \frac{\partial F_1(v,r,z)}{\partial z} - \frac{1}{r} \frac{\partial}{\partial r} \left[ D(r) \frac{\partial F_1(v,r,z)}{\partial r} \right] = \int_0^\infty \psi(r) \Gamma(v',r) G(v,r|v') \, dv' \quad (8)
\]

where

\[
G(v,r|v') = \frac{v(v,r)}{v'} \int_0^\infty \psi(v'',r) P(v'',r|v') \, dv''
\]

The right-hand side need contain only a single term to account for the source and sink of particles with size range \((0, v)\).

The second feature for our solution methodology is the property of similar breakage as found by Narasimhan et al.\(^{13}\) and Sathyagal et al.\(^{1,2}\) which is expressed by

\[
G(v,r|v') = g \left( \frac{\Gamma(v,r)}{\Gamma(v',r)} \right)
\]

For breakage frequencies of the degenerate form as represented in eq 5, eq 8 becomes

\[
G(v,r|v') = g \left( \frac{\Gamma(v)}{\Gamma(v')} \right)
\]

A fresh fit of the data of Sathyagal et al.\(^{1}\) yielded the following expression for the function \(G\):

\[
G(v|r|v') = a \left( \frac{\Gamma(v)}{\Gamma(v')} \right) - b \left( \frac{\Gamma(v)}{\Gamma(v')} \right)^{2.5} \quad (9a)
\]

where \(a = 1.6214\) and \(b = 0.6214.\)

### 4. Solution of the PBE

We address here the solution of eq 2 or eq 7, in which the breakage frequency is as identified in eqs 5 and 6 and the daughter drop size distribution is as in eq 9. The solution was attempted by two methods. First, we attempted a direct numerical solution using discretization and the control volume approach, which has the advantage of being general and not dependent on the use of the form of eq 2 but the disadvantage of having large computational times and memory requirements. Second, we demonstrate how the spectral expansion of the self-adjoint, diffusion linear operator can obtain the solution of eq 7 in a remarkably short time. Detailed account of linear operator methods is out of the scope of the present paper. The reader is referred to Ramkrishna and Amundson\(^{19}\) for this purpose.

For both of the solution methods, it is required that the fully developed turbulent flow be available at first hand. In view of this, the turbulent flow in the pipe was simulated using the low Reynolds number \(k-\epsilon\) model of Lai and So\(^{13}\) because of its proven superiority in predicting the flow characteristics near the wall.\(^{16}\)

#### 4.1. Direct Solution by Discretization and the Control Volume Approach

The direct approach toward solving eq 2 is to integrate the equation in space using the control volume approach in conjunction with any suitable method to treat the integral terms arising out of the breakage. Here, we use the discretization method of Kumar and Ramkrishna\(^{17}\) to discretize the internal coordinate using a geometric grid. Equation 2 after integration in the internal coordinate using \(M\)
geometric bins yields a set of $M$ scalar transport equations for the number density as

$$U_z(r) \frac{\partial \bar{N}(r,z)}{\partial z} - \frac{1}{r} \frac{\partial}{\partial r} \left( r D(r) \frac{\partial \bar{N}(r,z)}{\partial r} \right) = \sum_{k=1}^{M} n_{tk} \nu(v') P(v|v') \Gamma_k(r,z) \bar{N}_k(r,z) - \Gamma_z(r,z) \bar{N}(r,z)$$

We conserve here the first two moments (zeroth and first) of the drop size distribution where the redistribution coefficients are given by

$$n_{tk} = \int_{v_i}^{v_{i+1}} \frac{1}{v_{i+1} - v_i} \nu(v') P(v|v') \, dv + \int_{v_{i-1}}^{v_i} \frac{1}{v_i - v_{i-1}} \nu(v') P(v|v') \, dv$$

where the function $\nu(v') P(v|v')$ can easily be derived for the known function $G$ (eq 9a) as

$$\nu(v') P(v|v') = \frac{v'}{v} \left( \frac{\Gamma(v)}{\Gamma(v')} \right) \left\{ a - 2.5b \left[ \frac{\Gamma(v)}{\Gamma(v')} \right]^{1.5} \right\}$$

where $\Gamma(v)$ is the derivative of $\Gamma(v)$. The set of $M$ transport equations was integrated over the spatial domain of grid size 200 x 2000 using the control volume formulation of Patankar.\(^{18}\) The resulting system of discretized algebraic equations was solved using iterative procedure until the maximum normalized residue was less than $10^{-3}$ using the boundary conditions given by eq 4 for the case of uniform monodispersed drop size distribution over the pipe inlet.

4.2. Solution by Linear Operator Expansion. To enable solution by this method, we rewrite eq 7 by dividing it by $\Gamma_p(r) / R$, where $\Gamma_p(r) = \frac{2000}{r^2} r e^{r^2} dr$. Defining the following transformation of variables

$$t = \frac{z}{U(z)} \left( \frac{\Gamma_p(r)}{R} \right), \quad \varphi(r) \equiv \frac{\Gamma_p(r)}{R}, \quad F(v,r,t) \equiv \frac{1}{F_0} F_1(v,r,z)$$

eq 7 becomes

$$\frac{\partial}{\partial t} F(v,r,t) - \frac{1}{r \varphi(r)} \frac{\partial}{\partial r} \left[ r D(r) \frac{\partial}{\partial r} F(v,r,t) \right] = \Gamma_p \int_{v_0}^{\infty} \Gamma(v') G(v|v') \, dF(v',r,t)$$

(10)

with the “initial” condition

$$F(v,r,0) = F_0(v) \varphi(r), \quad F_0(v) = \frac{1}{F_0} \int_{v_0}^{\infty} v f_{1,0}(v') \, dv'$$

(11)

Equation 10 is now the focus of the solution methodology using the spectral expansion of the diffusion operator

$$L = - \frac{1}{r \varphi(r)} \frac{\partial}{\partial r} \left[ r D(r) \frac{\partial}{\partial r} f(r) \right], \quad D(L) \equiv \{ u \in \mathcal{H}; Lu \in \mathcal{H}; u'(0) = u'(R) = 0 \}$$

where $\mathcal{H} = L^2([0,R];r e^{r^2})$. For a clear understanding of the foregoing operator notation, the reader is referred to the work by Ramkrishna and Amundson.\(^ {19}\) The operator $L \equiv \{ L, D(L) \}$ is clearly a self-adjoint operator with nonnegative eigenvalues including zero as an eigenvalue (for a constant eigenfunction).

Because the eigenvalue problem cannot be solved analytically, the solution was obtained numerically using an Arnoldi algorithm (Saad\(^ {20}\)) with the aid of Matlab. For the present, we denote the eigenvalues by $\{ \lambda_j \}_{j=1}^{\infty}$, with $\lambda_1 = 0$, $\lambda_j > 0$, $j = 2, 3, \ldots$, with the corresponding eigenvectors $\{ \mathbf{z}_j \}_{j=1}^{\infty}$, which form an orthonormal basis in $\mathcal{H}$ with the property

$$\langle \mathbf{z}_j, \mathbf{z}_k \rangle = \delta_{jk}, \quad \langle \mathbf{u}, \mathbf{v} \rangle \equiv \int_0^R r \varphi(r) u(r) v(r) \, dr, \quad \mathbf{u}, \mathbf{v} \in \mathcal{H}$$

We construct the solution to the following recast of eq 10:

$$\frac{\partial}{\partial t} \mathbf{F}(v,t) + L \mathbf{F}(v,t) = \int_{v_0}^{\infty} \Gamma(v') G(v|v') \, d\mathbf{F}(v',t), \quad \mathbf{F}(v,0) = F_0(v) \mathbf{\psi}$$

(12)

where $\mathbf{F}(v,t) = \{ F(v,r,t) \}$ is the cumulative volume fraction redefined in terms of the transformed variables. The solution can be obtained as an expansion in terms of the eigenfunctions of $L$ as follows.

$$\mathbf{F}(v,t) = \sum_{j=1}^{\infty} F_j(v,t) \mathbf{z}_j, \quad F_j(v,t) \equiv \langle \mathbf{F}(v,t), \mathbf{z}_j \rangle$$

From eq 12, we can readily obtain

$$\frac{\partial F_j(v,t)}{\partial t} + \lambda_j F_j(v,t) = \int_{v_0}^{\infty} \Gamma(v') G(v|v') \, dF_j(v',t), \quad F_j(v,0) = F_0(v) \beta_j$$

(13)

where $\beta_j = \langle \mathbf{\psi}, \mathbf{z}_j \rangle$. Equation 13 may be rewritten as

$$\frac{\partial F_j(v,t)}{\partial t} = \int_{v_0}^{\infty} \Gamma(v') G(v|v') \, d\bar{F}(v',t), \quad \bar{F}(v,0) = F_0(v)$$

(14)

where $\bar{F}(v,t) = (1/\beta_j) F_j(v,t) e^{\lambda_j t}$. The transformed function $\bar{F}(v,t)$ is stripped of its dependence on $j$ because it satisfies eq 14, which is free of $j$. A very interesting aspect of eq 14 is that it represents the equation for evolution of the size distribution of particles by pure breakage in a well-mixed batch system. This equation can be readily solved by discretization or other methods. We present here the solution by the discretization method of Kumar and Ramkrishna.\(^ {17}\) Once the solution for $\bar{F}(v,t)$ is obtained, the solution for the function $\mathbf{F}(v,t)$ is immediate, for

$$\mathbf{F}(v,t) = \bar{F}(v,t) \sum_{j=1}^{\infty} \beta_j e^{-\lambda_j t} \mathbf{z}_j$$

(15)

We shall compare the solution (15) with that obtained by the direct numerical solution using the control volume approach in section 4.1.

5. Comparison of Solutions

It is of interest in this section to compare the solution of the population balance equation (7) using two different methods both in regards to the accuracy and the evolution of the drop size distribution.

To obtain the exact solution, the PBE for the well-mixed batch system is solved following the discretization method of Kumar and Ramkrishna,\(^ {17}\) conserving zeroth
and first moments of drop size distribution to obtain a set of coupled ordinary differential equations.

\[
\frac{dN_i(t)}{dt} = \sum_{k=1}^{M} \Gamma_{i,k} N_k(t) - \Gamma_i N_i(t) \quad (16)
\]

The set of simultaneous first-order differential equations for the droplet number densities (in space) was integrated using a Runge–Kutta fourth-order integration technique with the monodispersed drop size distribution. The droplet volume was discretized using a geometric grid with the number of classes \(M\) equal to 50 and a consecutive volume ratio \(v_{i+1}/v_i\) of 1.1. It requires here the mention of the fact that the grid used for the discretization of the drop volume was the same as that used in the direct solution approach. The exact solution was constructed using the first 70 eigenvalues and eigenvectors and the number density evolution obtained after simulating the PBE for the well-mixed batch system.

The comparison of the drop size distribution due to the exact solution given by eq 15 with that obtained with the detailed numerical solution, at two radial locations \(r/R = 0.96\) and \(z/R = 263\), is shown in Figures 2–5. It should be noted that the locations at which the comparison is presented are chosen so as to represent the evolution of the drop size distribution at both the radial and axial locations that correspond to significantly different transformed times.

A foremost observation of all of the figures to be made is that the exact solution methodology performs as well as the direct simulation approach but with significantly smaller computational effort. The comparison of the computational times is discussed at a subsequent stage.

Careful observation of Figures 2 and 3 leads to the clear inference that the peak number density at a radial location of \(r/R = 0.96\) corresponds to 55% of the original size, while the peak number density at a radial location of \(r/R = 0.98\) shifts to significantly smaller droplet volume (45%). It is interesting to note that a minor change of the radial position (from 0.96 to 0.98) for a given axial location leads to a significantly different drop size distribution. This is justifiable because the energy dissipation rate changes with a steep gradient near the wall, which leads to larger breakage rates. In addition, the time spent by the drops at a given axial location in the breakage environment is larger near the wall because of lower velocities that further contribute to an overall increase in breakage. (Of course, the diffusion of a drop toward the pipe wall progressively slows down its progress along the axial direction.) The effects of both the energy dissipation rate and velocity are clearly reflected in the transformed time, which is a strong nonlinear function introduced in terms of the radial location. Figures 4 and 5 show the drop size distributions at two different axial locations at the same radial location. As an obvious matter of fact, the presence of fine drops increases in the axial direction at the same radial location. It is of interest here to note that the change in the drop size distribution introduced
because of the large change in the axial position is relatively smaller compared to the one introduced to the change in the radial position at a given axial location. This can be readily understood as a consequence of the transformed time, which scales linearly with the axial position versus its strong nonlinear variation with respect to the radial location introduced because of radial variation of the flow characteristics.

It is of interest to mention here that all of the computations were performed on a Sun E6500 server. The codes were run on a single CPU without the use of parallel programming. A comparison of the CPU times is given in Table 1. The time for the calculation of number densities using the batch solution that involves finding the transformed time and the summation term on the right-hand side of eq 15 is denoted by $T$, and it is of the order of seconds and scales linearly with the number of locations at which it is to be calculated. It is clear that the savings in computational time can be achieved by an order of magnitude by using the exact solution. It should be noted that the times can be drastically reduced using the faster processors but retaining the relative differences in computational times for the two different methods.

In summary, the exact solution is able to predict the drop size distribution as well as the direct solution approach, which generally takes enormous computational resources and time.

6. Self-Similar Solutions

Because of the fact that $F(v, t)$ is known to have a self-similar behavior with respect to the similarity variable, $\eta = D \sqrt{t} / \sqrt{v}$, the solution (15) must clearly display self-similarity properties. Thus, $F(v, r, t) \sum_{n=1}^{\infty} \beta_n g^{-2} t_x(r)$ must display self-similarity when plotted versus the similarity variable $\eta$. To illustrate the self-similarity, the number densities obtained using the direct solution of the PBE at various spatial locations were transformed according to eq 14 and plotted against the similarity variable (Figure 6a). It can readily be seen that the transformed number densities collapse on a single curve after sufficiently long times. For a clear illustration, the plot is presented with the transformed number densities corresponding to the larger transformed times (Figure 6b) at which self-preserving behavior is approached.

This is a remarkable result because the self-preserving nature of the drop size evolution in a fully developed turbulent flow can be used to extract the breakage function from the experimental measurements using the inverse problem approach.

7. Conclusions

1. A remarkably simple exact solution for the PBE for the case of pure breakage in fully developed pipe flow has been presented using the linear operator theory.

2. The proposed solution method yields a solution to the complex PBE for the case of pure breakage in a fully developed turbulent pipe flow from the solution of the PBE for the well-mixed batch system.

3. Estimates of the drop size distribution obtained using the exact solution are as good as the predictions from the rigorous computational method but at substantially reduced computational cost.

4. It is shown that upon simple transformation the PBE for the pure breakage in a fully developed turbulent pipe flow that involves diffusive transport also exhibits the self-similar solution.

Nomenclature

$D$ = droplet diffusivity, m$^2$/s
$f_i$ = number density (in particle and spatial volumes), m$^{-3}$m$^{-3}$
$f_{i,0}$ = initial number density as a function of the drop volume, m$^{-3}$m$^{-3}$
$F_0$ = initial cumulative volume fraction
$F_1$ = cumulative volume fraction
$G(v')$ = cumulative distribution function for the particles of size $v'$ formed because of the breakage of particles of volume $v'$
$H$ = Hilbert space
$i, k =$ indices for the drop volume
$k =$ turbulent kinetic energy, m$^2$/s$^2$
$\mathcal{L}$ = linear subspace
$M =$ number of geometric bins
$n_{i,k} =$ redistribution coefficients in the discretization method of Kumar and Ramkrishna$^{17}$

<table>
<thead>
<tr>
<th>Table 1. Details of the CPU Time for the Simulations</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>task</strong></td>
</tr>
<tr>
<td>-----------------------------------------------</td>
</tr>
<tr>
<td>flow simulation</td>
</tr>
<tr>
<td>estimation of eigenvalues and eigenfunctions</td>
</tr>
<tr>
<td>simulation of a well-mixed batch PBE</td>
</tr>
<tr>
<td>direct solution with the control volume approach</td>
</tr>
<tr>
<td>total time required for an exact solution</td>
</tr>
</tbody>
</table>

$T$ denotes the time for the calculation of the number densities (of the order of seconds) using the batch solution, which involves finding the transformed time and the summation term on the right-hand side of eq 15.
\( N = \text{number density in space, m}^{-3} \)
\( P(u|v') = \text{distribution function for the particles of size } v' \text{ formed because of the breakage of particles of volume } u' \)
\( r = \text{radial distance from the center of the pipe, m} \)
\( R = \text{radius of the pipe, m} \)
\( t = \text{time, s} \)
\( T = \text{time for the calculation of the number densities from a batch solution, min} \)
\( U_i = \text{mean axial velocity, m/s} \)
\( U = \text{mean velocity vector, m/s} \)
\( v = \text{particle volume, m}^3 \)
\( v_0 = \text{initial drop size, m}^3 \)
\( z = \text{axial distance, m} \)
\( z = \text{eigenvectors} \)

**Greek Letters**

\( \psi = \text{spatial initial size distribution function} \)
\( \Gamma = \text{breakage rate, s}^{-1} \)
\( \Gamma_R = \text{breakage frequency function with spatial dependence} \)
\( \Gamma_v = \text{breakage frequency function with particle space dependence} \)
\( \epsilon = \text{energy dissipation rate, m}^2/s^3 \)
\( \eta = \text{similarity variable} \)
\( \lambda = \text{eigenvalues} \)
\( \mu_M = \text{kinetic viscosity, kg/ms} \)
\( \mu_t = \text{turbulent viscosity, kg/ms} \)
\( v = \text{number of daughter drops formed as a result of the breakage of drop of volume } v \)
\( q(r) = F(r)/\Gamma_R \)
\( \Phi = \text{volume fraction} \)
\( \rho = \text{fluid density, kg/m}^3 \)

**Literature Cited**


