ANALYSIS OF AXIALLY DISPERSED SYSTEMS WITH GENERAL BOUNDARY CONDITIONS—III

SOLUTION FOR UNMIXED AND WELL-MIXED APPENDED SECTIONS

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Abstract—The axial dispersion problem formulated in Part I[1] of this series is solved for cases 2-4 and 7 identified therein. The cases analyzed here consider appendages to the finite axially dispersed reactor which are either semi-infinite sections of finite dispersion or well-mixed sections of finite capacity. The findings possess features akin to those found in Part II[7] for case 1 but obviously displaying variations in detail.

1. INTRODUCTION

In a previous paper [1], the authors had formulated the problem of axial dispersion in tubes of finite length for situations wherein there is non-zero dispersion outside of the tube. The tube proper was viewed as being appended by an upstream or fore section and by a downstream or aft section. A linear rate process such as a first order, isothermal, chemical reaction occurring in the tube proper was considered to take place in the entire extended tube with a kinetic coefficient which has a uniform value in the tube proper and vanishes outside of it. The formulation presented in Part I[1] overcomes the constraint of the use of the Danckwerts boundary conditions for the transient analysis of finite axially dispersed systems. A host of boundary conditions were suggested in Part I[1] (depending on the extent of dispersion in the appended sections) of which the Danckwerts boundary conditions are a special case. Familiarity with the contents of this previous paper[1] is essential for an understanding of the material of this paper. In this paper, we will investigate in detail what were identified as case 2 \((0 < D_+ < \infty, D_- = 0)\), case 3 \((0 < D_+ < \infty, D_- = \infty)\), case 4 \((D_+ = 0, 0 < D_- < \infty)\) and case 7 \((D_+ = \infty, 0 < D_- < \infty)\) in [1]. \(D_+\) and \(D_-\) denote the dispersion coefficients in the fore and aft sections. In terms of the dimensionless variables defined in [1], the conservation equation in dimensionless form for a first order, isothermal, chemical reaction is

\[
\frac{1}{r(x)} \left[ \frac{-\delta^2 Y(x)}{\delta x^2} + Pe \frac{\delta Y(x)}{\delta x} + Da Y(x) \frac{\delta}{\delta x} + Da Y(x) \right] - Da Y(x) = - \frac{\delta Y(x)}{\delta x} \quad a < x < b, \quad \tau > 0
\]

where

\[
Y(x) = \begin{cases} 0, & |x| \geq 1 \\ 1, & |x| < 1 \end{cases}
\]

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to note that the Danckwerts boundary condition may be obtained at either end of the tube proper by allowing $g_-$ or $g_+$ to vanish (i.e., by allowing the volume of the stirred tank preceding or succeeding the tube proper to vanish). The initial condition for the transient problem is stated as

$$
\chi(x, 0) = F(x), \quad a < x < b
$$

$$
\chi_+(0) = F_+.
$$

2. SOLUTION OF PROBLEM

The formulation of the transient axial dispersion problem presented here is amenable to use of the spectral theory of singular second order differential operators. We denote the general differential operator in cases 2, 3, 4 and 7, as $L = \{L, D(L)\}$ where $L$ is a differential or matrix-differential expression and $D(L)$ is the set of functions on which $L$ is supposed to act. $D(L)$ is the subspace of the Hilbert space $\mathcal{H}$. For cases 2, 3, 4 and 7, we define $L, D(L)$, $\mathcal{H}$ and the inner product in $\mathcal{H}$ as follows. We denote $\mathcal{L}_2(a, b, r(x) e^{-\rho x})$ as the space (set) of functions $f(x)$ such that

$$
\int_a^b r(x) e^{-\rho x} |f(x)|^2 \, dx < \infty.
$$

Case 2. $0 < Pe_- < \infty, Pe_+ = \infty$; $L = A_-$. $A_- = \frac{1}{r(x)} \left[ -\frac{d^2}{dx^2} + Pe \frac{d}{dx} + DaY(x) \right]$, $-\infty < x \leq 1$.

$\mathcal{H} = \mathcal{L}_2(-\infty, 1, r(x) e^{-\rho x})$.

$D(A_-) = \{u - (u(x) - 1 \leq x < \infty): u \in \mathcal{H}, A_-u \in \mathcal{H}, u'(1) = 0, u(x) and u'(x) locally absolutely continuous in (-\infty, 1]\}$.

$$
\langle f, u \rangle = \int_{-\infty}^1 r(x) e^{-\rho x} f(x) u^*(x) \, dx \quad \forall f, u \in \mathcal{H}.
$$

$u^*(x)$ is the complex conjugate of $u(x)$.

Case 3. $0 < Pe_- < \infty, Pe_+ = 0$; $L = A^0_+$. $A^0_+ = \frac{1}{r(x)} \left\{ \frac{d^2}{dx^2} + Pe \frac{d}{dx} + DaY(x) \right\} 0$, $-\infty < x \leq 1$.

$\mathcal{H} = \mathcal{L}_2(-\infty, 1, r(x) e^{-\rho x}) \oplus \mathcal{C}$.

The Hilbert space $\mathcal{H}$ in this case is the direct sum[2] of $\mathcal{L}_2(-\infty, 1, r(x) e^{-\rho x})$ and the complex number space $\mathcal{C}$. We represent an element of $\mathcal{H}$ as

$$
f = \begin{bmatrix} f(x) \\ f_+ \end{bmatrix}, \quad -\infty < x \leq 1.
$$

The domain of $A^0_+$ and the inner product in $\mathcal{H}$ are defined as

$$
D(A^0_+) = \{f: f \in \mathcal{H}, A^0_+f \in \mathcal{H}, f(1) = f_+, f(x) and f'(x) locally absolutely continuous in (-\infty, 1] \}$.

$$
\langle f, u \rangle = \int_{-\infty}^1 r(x) e^{-\rho x} f(x) u^*(x) \, dx + g_1 f_+ u_+ e^{-\rho x}.
$$

The operator $L (L = A_-, A_+, A^0_+, and A^0_0)$ is shown to be self-adjoint in the Appendix. The eigenvalues of
L are therefore real. The behavior at \( x - x_{\infty} \) of the functions belonging to \( D(L) \) is somewhat different from conditions (4) and (6). While condition (4) is automatically implied when \( \chi \in D(Y) \) with \( \chi \) and \( Y \) being defined in eqns (11) and (12) for \( \Phi_{-} \) and in eqns (15) and (16) for \( \Phi_{+} \), condition (6) implies that \( \chi \in D(Y) \) with \( \chi \) and \( Y \) being defined in eqns (20) and (21) for \( \Phi_{+} \) and in eqns (24) and (25) for \( \Phi_{+} \). The solutions obtained using the conditions in (12), (17), (21) and (26) satisfy the physical requirements (4) and (6) however. With the help of the definitions given earlier, the transient problems for cases 2, 3, 4 and 7 of [1] can be most compactly cast as

\[
L \chi(x) - S(t) = -\frac{d}{dx} \chi(x) \quad (28)
\]

\[
\chi(0) = F \quad (29)
\]

where

\[
S(t) = \left\{ \begin{array}{l}
S(x, t) = \frac{D_u Y(x)}{r(x)}; \\
S_+(x, t) = \left[ \begin{array}{c}
\frac{D_u Y(x)}{r(x)} \\
0
\end{array} \right]
\end{array} \right.
\]

\[
a < x < b, \tau > 0 \quad \text{for} \ L = \Phi_{-}, \Phi_{+} \quad (30)
\]

\[
S(t) = \left\{ \begin{array}{l}
S(x, t) = \left[ \begin{array}{c}
\frac{D_u Y(x)}{r(x)} \\
0
\end{array} \right]
\end{array} \right.
\]

\[
a < x < b, \tau > 0 \quad \text{for} \ L = \Phi_{-}^2, \Phi_{+}^2. \quad (31)
\]

The time-dependence in \( S(t) \) is inserted to accommodate other transient source terms in addition to those in eqn (1). \( F \) is the vectorial representation of the initial concentration distribution.

In our earlier paper [1], we had suggested two ways of obtaining the solution of the boundary-initial value problem. The two ways are equivalent in that they are associated with the spectral properties of the differential operator \( L \). The first is via the method of integral transforms. In order to understand the solutions presented in this paper, it is not necessary to be familiar with the method of development of these transforms. The second approach, the spectral theoretic approach, directly exploits the spectral representation of the operator \( L \) and functions of it and establishes the background of the integral transforms. Symbolic formulae for the transform and its inverse and the properties of the transform which are useful in obtaining the solution of the boundary-initial value problem were given in Part I [1]. We call the general differential operator \( L \) as the axial dispersion operator. It was mentioned in Part I [1] that the spectrum (set of eigenvalues) of \( L \) in general consists of continuous eigenvalues and discrete eigenvalues.

For cases 2 and 4, the transforms of \( \chi(\tau) \) with respect to continuous and discrete eigenvalues are defined as follows.

\[
\bar{\chi}(\omega, \tau) = \int_a^b r(x)W(\omega, x)\chi(x, \tau)e^{-\omega x} dx,
\]

\[
0 < \omega < \infty \quad (32)
\]

\[
\bar{\chi}(\tau) = \int_a^b r(x)W(\tau, x)\chi(x, \tau)e^{-\tau x} dx
\]

\[
= \langle \chi(\tau), W_\tau \rangle. \quad (33)
\]

\( W(x, \omega) \) is the normalized eigenfunction of \( L \) corresponding to its continuous eigenvalues and \( W_\tau(x) \) is the normalized eigenfunction of \( L \) corresponding to its discrete eigenvalue \( \lambda \). For cases 3 and 7, we define the transforms of \( \chi(\tau) \) with respect to continuous and discrete eigenvalues as follows.

\[
\bar{\chi}(\omega, \tau) = \int_a^b r(x)W(\omega, x)\chi(x, \tau)e^{-\omega x} dx
\]

\[
+ g W(\omega, \tau)\chi(x, \tau)e^{-\omega x}, \quad 0 < \omega < \infty \quad (34)
\]

\[
\bar{\chi}(\tau) = \int_a^b r(x)W(\tau, x)\chi(x, \tau)e^{-\tau x} dx
\]

\[
+ g W(\tau, \tau)\chi(x, \tau)e^{-\tau x} \quad (35)
\]

c and \( g \) are defined in Table 1 for various cases of interest to the present work. \( \chi_+ (\tau) \) is \( \chi_+ (\tau) \) for case 3 and \( \chi_-(\tau) \) for case 7.

A crucial property of the integral transforms defined in eqns (32)-(35) which is very useful in obtaining the solution to the transient problem defined in eqns (28) and (29) is

\[
\overline{(Lf)}(\omega) = \left( \omega^2 + \frac{Pe^2}{4} \right) Pe \overline{f(\omega)} \forall f \in D(L)
\]

Using the above, \( \bar{\chi}(\omega, \tau) \) and \( \bar{\chi}_\tau \) are obtained as [1, 3]

\[
\bar{\chi}(\omega, \tau) = F(\omega)e^{-\omega x} + \int_0^\tau e^{-\omega(t-r)}S(\tau) d\tau \quad (36)
\]

\[
\bar{\chi}(\tau) = F(\tau)e^{-\tau x} + \int_0^\tau e^{-\tau(t-r)}S(\tau) d\tau. \quad (37)
\]

<table>
<thead>
<tr>
<th>Table 1</th>
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<tbody>
<tr>
<td>Case</td>
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Theoretical approach directly exploits the spectral representation of the operator \( L \) and functions of it and establishes the background of the integral transforms.
The solution in cases 2 and 4 is recovered from the integral transforms (36) and (37) using the inversion formula

$$\chi(x, \tau) = \sum_{j=1}^{N} \tilde{\chi}_j(\tau) W_j(x) + \int_{\tau}^{\infty} \tilde{\chi}(\tau, \tau) W(x, \omega) d\omega \quad (38)$$

$N$ is the number of discrete eigenvalues. In cases 3 and 7, the solution is obtained using the inversion formula

$$\chi(x, \tau) = \sum_{j=1}^{N} \tilde{\chi}_j(\tau) \begin{bmatrix} W_j(x) \\ W_j(\omega) \end{bmatrix} + \int_{\tau}^{\infty} \tilde{\omega}(\omega, \tau) \begin{bmatrix} W(x, \omega) \\ W(\omega, \omega) \end{bmatrix} d\omega. \quad (39)$$

The continuous eigenvalues of $L$ (without $A$, and $A^\pm$) lie in the interval $[P e P e_0/4, \infty]$ [4, 5] with $P e_0$ being defined in Table 1 for various cases. The necessary condition for existence of at least one discrete eigenvalue of $L$ is [4, 5]

$$\frac{P e P e_0}{4} > \min \left\{ \frac{P e^2}{4} + D a, \frac{P e}{2} \right\}. \quad (40)$$

When condition (40) is satisfied, the discrete eigenvalues, if they exist, lie in the interval

$$\frac{P e P e_0}{4} \geq \min \left( \frac{P e^2 + D a}{2 P e}, \frac{P e P e_0}{4} \right) \quad [4, 5].$$

For cases 2 and 3, the eigenfunction $W(x, \omega)$ corresponding to the continuous eigenvalues has the expression [4, 5]

$$W(x, \omega) = \left[ \omega \left[ \alpha \cos (2 \alpha x) + \omega_1 \sin (2 \alpha x) \right] \cos \left( \omega(x + 1) \right) - \alpha \sin (\omega(x + 1)) \right] \frac{P e}{P e_0} e^\left( -1 \right) e^{P e/2} \frac{\omega_1}{\omega} e^{2P e} \left( \omega(x - 1) \right), \quad |x| \leq 1 \quad (41)$$

where

$$\omega = \sqrt{\left[ \frac{P e}{P e_0} - \frac{P e^2}{4} \right]}$$

(42a)

$$\omega = \sqrt{\frac{P e}{P e_0} + \frac{P e^2}{4} - \frac{P e}{2} - D a} \quad (42b)$$

$$\omega = \frac{P e}{2} - \frac{h}{g} \left( \omega^2 + \frac{P e^2}{4} \right) \quad (42c)$$

with $\omega, \omega_1, \alpha$ and $\vartheta(\omega)$ being defined in (42).

For cases 2 and 3, the eigenfunction $u(x)$ corresponding to the discrete eigenvalue $\lambda_j$ has the expression [4, 5]

$$u(x) = p_j(x) \exp \left( \frac{P e}{2} x \right), \quad -\infty < x \leq 1$$

(43)

$$p_j(x) = \exp (f_j x), \quad -\infty < x \leq 1$$

(44)

$$e_j = \sqrt{\lambda_j - \frac{P e^2}{4} - D a}$$

(45)

and

$$f_j = \sqrt{\frac{P e^2}{4} - \lambda_j \frac{P e_0}{2}}$$

(46)

where $e_j$ and $f_j$ are defined as

$$e = \frac{h_j \tan (e_j) + e_j r_j}{m_j \tan (e_j) + e_j (1 - \tan^2 (e_j))}$$

(47)

$$h_j = \frac{P e_0}{2} - g \lambda_j$$

(48)

$$\alpha, \beta, \gamma, \alpha^\pm, A^\pm, A^\pm$$

(49)

The constant $a_j$ and $b_j$ are expressed as

$$a_j = (h_j \tan (e_j) + e_j r_j), \quad b_j = (e_j \tan (e_j) - h_j) r_j$$

(50)

The discrete eigenvalues of $L$ (without $A$, and $A^\pm$) and

$$\vartheta(\omega) = \frac{P e}{P e_0} \varphi(\omega),$$

(51)

where $\varphi(\omega)$ is the characteristic equation

$$\tan (2 \varphi) = 0$$

(52)

$e, f$ and $h$ are defined as in eqns (45) and (47) with $\lambda$ replacing $\lambda_j$. The higher the value of a discrete eigenvalue, the higher is the value of $j$. The smallest discrete eigenvalue is denoted $\lambda_1$.

For cases 4 and 7, the eigenfunction $u(x)$ has the form [4, 5]

$$u(x) = p_j(x) \exp \left( \frac{P e}{2} x \right), \quad -1 < x < \infty$$

(53)

$$p_j(x) = \begin{cases} a_j \cos (e_j x) - b_j \sin (e_j x), & -1 < x < 1 \\ \exp (-f_j x), & 1 < x < \infty \end{cases}$$

(54)

$$W(x, \omega) = \left[ \omega \left[ \alpha \cos (2 \alpha x) + \omega_1 \sin (2 \alpha x) \right] \cos \left( \omega(x + 1) \right) - \alpha \sin (\omega(x + 1)) \right] \frac{P e}{P e_0} e^{2P e} \left( \omega(x - 1) \right), \quad |x| \leq 1$$

(55)

with $\omega, \omega_1, \alpha$ and $\vartheta(\omega)$ being defined in (42).
with $a$, $b$, $c$ and $f$ being defined as in eqns (45) and (46). The discrete eigenvalues are the roots of the characteristic equation (48). The normalized eigenfunction $W(x)$ for various cases is then defined as

$$W(x) = \frac{u(x)}{\sqrt{(u, u)}} \quad a < x < b$$  

For details of: (a) the identification of the spectrum (set of continuous and discrete real-valued eigenvalues); (b) derivation of the characteristic equation for discrete eigenvalues; and (c) formulation of the normalized eigenfunctions; the reader is referred to some of our other works[&6].

From a spectral-theoretic viewpoint, the solution of eqn (28) subject to the initial condition (29) may be expressed as[1]

$$\chi(t) = e^{-tL}F + \int_0^t e^{-t-(t')}S(t') \, dt'.$$  

An important property of the operators defined earlier is the representation of $L$ and functions of it in terms of the orthogonal projections associated with the continuous and discrete eigenvalues of $L$. For example, the representation for $e^{-tL}$ is

$$e^{-tL} = \sum_{n=0}^{\infty} \frac{(-t)^n}{n!} P_n.$$  

For cases 2 and 4, the differential projection $dP(\lambda)$ corresponding to the continuous eigenvalue $\lambda$ of $L$ is defined for any $u \in \mathcal{H}$ as

$$dP(\lambda)u = d\omega \left\{ W(x, \omega) \int_a^b r(y) W*(y, \omega) \right. \times \left. e^{-P_0 u(y)} \, dy \right\}.$$  

For cases 3 and 7, for any $u \in \mathcal{H}$, $dP(\lambda)$ has the expression

$$dP(\lambda)u = d\omega \left\{ \int_a^b r(y) e^{-P_0 u(y)} W*(y, \omega) \, dy + g u W*(c, \omega) e^{-P_0} \right\}$$  

$$\begin{bmatrix} W(x, \omega) \\ W(c, \omega) \end{bmatrix}$$  

$a$, $b$, $c$ and $g$ for various cases are defined in Table 1. $u$ is $u_+$ for case 3 and $u_-$ for case 7. The projection $P_j$ corresponding to the discrete eigenvalue $\lambda_j$ of $L$ is defined as

$$P_j u = \langle u, W_j \rangle W_j \forall u \in \mathcal{H}.$$  

Theorems which are useful in the spectral analysis of the axial dispersion operators arising in cases 2, 3, 4 and 7 are stated in the Appendix. In view of eqns (52), (53) and (55), one can show that the solution (51) is equivalent to solution (38) in cases 2 and 4. Similarly, with the help of eqns (52), (54) and (55), it can be deduced that solution (51) is equivalent to solution (39).

It is interesting to note that the expressions for the eigenfunctions in case 2, eqns (41), (42), (44)-(48) can also be obtained from eqns (1.1)-(1.4) and eqns (1.12)-(1.15) of part II[7] of this series with the substitution $Pe_+ = \infty$. Similarly, the expressions for the eigenfunctions in case 4, eqns (42), (43) and (45)-(49), can be obtained from eqns (1.1)-(1.3), (1.5) and (1.12)-(1.15) of [7] for the special case $Pe_+ = \infty$.

The existence of discrete eigenvalues depends on the relative magnitudes of the parameters $Pe$, $Pe_0$, $g$ and $Da$. Compared to case 1, the discrete spectral behavior for cases 3 and 7 is much more complicated. We therefore recognize five different categories of these parameters. It's the discrete spectral behavior (and not the continuous spectral behavior) that is different in different categories.

3. SPECTRAL BEHAVIOR OF L

We investigate the behavior of the spectrum of the axial dispersion operator $L$ by dividing the entire range of parameters into five categories. These are defined below.

Category I. This category consists of parameters which are such that

$$\frac{Pe}{2g} \geq \frac{Pe_0}{2} > \frac{Pe^2}{4} + Da.$$  

Let

$$M = \text{smallest integer } \geq \frac{2}{\pi} \sqrt{P_0}$$  

where

$$P_0 = \frac{PePe_0}{4} - \frac{Pe^2}{4} - Da.$$  

If

$$\sqrt{P_0} > (M - 1) \frac{\pi}{2} + \frac{1}{2} \tan^{-1} \left\{ \frac{Pe}{2\sqrt{P_0}} \left( 1 - \frac{gPe_0}{2} \right) \right\}$$  

then there are $M$ discrete eigenvalues, otherwise there are $(M - 1)$ discrete eigenvalues. For the parameters lying in this category, a sufficient condition for existence of at least one discrete eigenvalue is obtained by substituting $M = 1$ in eqn (59). The discrete eigenvalues, if they exist, lie in

$$\left( \frac{Pe^2}{4} + Da, \frac{PePe_0}{4} \right).$$  

Category II. The parameters lying in this category satisfy the relations

$$\frac{PePe_0}{4} > \frac{Pe}{2g} > \frac{Pe^2}{4} + Da.$$  

(60)
If
\[ \sqrt{Pe} > \frac{\pi}{2} M - \frac{1}{2} \tan^{-1} \left( \frac{Pe}{2\sqrt{Pe_0} \left( g \frac{Pe_0}{2} - 1 \right)} \right) \] (61)
then \( L \) has \((M + 1)\) discrete eigenvalues. Otherwise, there are \(M\) discrete eigenvalues. Thus, for the parameters lying in this category, \( L \) has at least one discrete eigenvalue (since \( M \geq 1 \)).

**Category III.** The parameters belonging to this category satisfy the relations
\[ \frac{PePe_0}{4} > \left( \frac{Pe^2}{4} + Da \right) > \frac{Pe}{2g} \] (62)

If
\[ g \left( Da + \frac{Pe^2}{4} \right) - \frac{Pe}{2} \] (63)
and if
\[ \tanh \left[ 2 \sqrt{\left( \frac{Pe^2}{4} - \lambda_0 \frac{Pe}{Pe_0} \right)} \right] = \frac{2g \sqrt{\left( \frac{Pe^2}{4} - \lambda_0 \frac{Pe}{Pe_0} \right) + Pe}}{-2 - 2g \sqrt{\left( \frac{Pe^2}{4} - \lambda_0 \frac{Pe}{Pe_0} \right) + Pe}} \] (64)
when
\[ \lambda_0 = \frac{gPe - Pe}{2g^2} = \frac{Da}{1 - \frac{Pe}{Pe_0}} \]
then \( L \) has one discrete eigenvalue in
\( \left( \frac{Pe}{2g^2} + Da \right) \). Otherwise, there is no discrete eigenvalue in this interval. If
\[ g \left( Da + \frac{Pe^2}{4} \right) - \frac{Pe}{2} \] (65)
and if condition (61) is satisfied, then \( L \) has \((M + 1)\) discrete eigenvalues in
\( \left( \frac{Pe^2}{4} + Da, \frac{PePe_0}{4} \right) \).

If either condition (61) or condition (65) is not satisfied, then \( L \) has \( M \) discrete eigenvalues in the interval
\( \left( \frac{Pe^2}{4} + Da, \frac{PePe_0}{4} \right) \).

If both (61) and (65) are violated, then there are \((M - 1)\) discrete eigenvalues in
\( \left( \frac{Pe^2}{4} + Da, \frac{PePe_0}{4} \right) \).

The reader should note that
\[ \left\{ \frac{Pe^2}{4} + Da, \frac{PePe_0}{4} \right\} \] do not belong to the set of discrete eigenvalues.

**Category IV.** The parameters which satisfy the relation
\[ \min \left\{ \left( \frac{Pe^2}{4} + Da \right), \frac{PePe_0}{4} \right\} = \frac{PePe_0}{4} \] (66)
belong to this category. In this case, \( L \) has no discrete eigenvalues and thus its spectrum consists exclusively of continuous eigenvalues.

**Category V.** This category consists of parameters which are such that
\[ \left\{ \frac{Pe^2}{4} + Da, \frac{PePe_0}{4} \right\} \][estimate of the number of discrete eigenvalues.]

The formulae for the number of discrete eigenvalues of \( L \) for the parameters lying in categories I-IV were obtained by studying the nature of the l.h.s. and r.h.s. of eqn (48) for various \( \lambda \). For the parameters lying in category V, the denominator of the right side of eqn (48) exhibits a rather complex behavior. Therefore, we have been unable to derive a formula for \textit{a priori} estimate of the number of discrete eigenvalues.

For the parameters lying in the five categories defined earlier, the continuous eigenvalues of \( L \) \((L = \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)\) are always greater than or equal to \( PePe_0/4 \). We note here that in cases 2 and 4, all the parameters lie in categories I and IV. Condition (59) in cases 2 and 4 can be deduced from condition (1.17) of [7] for the special case of \( Pe_- \) or \( Pe_+ \) being infinite. One can verify that the spectrum of the axial dispersion operator \( \lambda_1 \) can also be obtained as the limit of the spectrum of the axial dispersion operator in case [7] as \( Pe_- \) becomes infinite. Similarly, the spectrum of \( \lambda_2 \) can be obtained as the limit of the spectrum of the axial dispersion operator in case [7] as \( Pe_+ \) becomes infinite. The reader interested in the details of the formulae for the number of discrete eigenvalues of \( L \) \((L = \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)\) is referred to some of our other works[4,5]. Table 2 shows how the number of discrete eigenvalues, \( N \), can be estimated in cases of interest to the present paper.
Having investigated the spectral behavior of the axial dispersion operators arising in cases 1–4 and 7 in Parts II and III of this series, we feel it proper here to pictorially represent the nature of the eigenvalues of the axial dispersion operator in each of the nine cases defined in Table 1 of Part I[1]. This representation is shown in Table 3. The discrete eigenvalues are represented by vertical lines and the continuous eigenvalues are represented as a band of arbitrary height extending horizontally.

The continuous spectrum in case 1 comprises two regions, region $A$ and region $B$. Depending on whether certain conditions are satisfied or not, there may be one or more discrete eigenvalues. As the larger of $Pe_-$ and $Pe_+$ increases, region $A$ expands and the left boundary of region $B$ shifts to the right. In the limiting case of infinite $Pe_-$ (case 4) or infinite $Pe_+$ (case 2), region $B$ vanishes and the continuous spectrum then consists only of region $A$. In case 2 ($Pe_+ = \infty$) or in case 4 ($Pe_- = \infty$), as $Pe_+$ is increased ($Pe$ and $Da$ being fixed), the left boundary of the continuous spectrum moves to the right while the number of discrete eigenvalues increases. In the limiting case when both $Pe_-$ and $Pe_+$ become infinite, the continuous spectrum of the axial dispersion operator vanishes and the eigenvalues are discrete in nature. The spectra of the dispersion operators in cases 3 and 7 exhibit similar behavior. The spectrum in case 3 consists exclusively of discrete eigenvalues in the limit of infinite $Pe_-$ (case 6). Therefore, the spectrum in case 6 is obtained as the limit of the spectrum in case 3 as $Pe_-$ becomes infinite. Similarly, the spectrum in case 8 (eigenvalues are exclusively discrete) is obtained as the limit of the spectrum in case 7 as $Pe_+$ is made infinite. Lastly, in case 9, the spectrum consists entirely of discrete eigenvalues.

4. APPLICATIONS

As was done in Part II of this series[7], the properties of the axial dispersion operators (arising in cases 2, 3, 4 and 7) described in the last section can be used to obtain solutions to transient problems arising in other similar situations in axially dispersed systems. We do not wish to provide here the details of formulation and solution of transient problems for various applications discussed in [7] for they are not significantly different from those provided elsewhere[5, 7]. In systems where there is a stirred tank of
known volume before or after the tube proper, the dispersion parameters in the tube proper and the partially mixed fore or aft section can be estimated by measuring the tracer concentration at two locations (instead of at three locations as in case 1 (7)). One of the measurement locations could be the stirred tank itself for it is much more convenient to measure the tracer concentration in the stirred tank than at any other location. If the tube proper is preceded and succeeded by stirred tanks (case 9) whose volumes are known, then measurements need only be made at one location to estimate the dispersion coefficient in the tube proper. An ideal measurement location in this case is the stirred tank placed after the tubular equipment.

5. RESULTS AND DISCUSSION

Analytical evaluation of integrals appearing in eqns (38) and (39) is ruled out in view of the complicated nature of the eigenfunctions corresponding to the continuous eigenvalues of the axial dispersion operator. The inversion of the integral transforms in cases 2, 3, 4 and 7 was therefore done numerically.

Some of the calculations done to identify the discrete eigenvalues of \( L \) in cases 2, 3, 4 and 7 are tabulated in Table 4. \( N \) denotes the number of discrete eigenvalues predicted using the chart in Table 2. The eigenvalues obtained by solving the characteristic equation (48) are shown in the last column of the table. The results of computations done for a first order, isothermal, chemical reaction are shown in Figs. 1–4. The transient predictions are compared with those obtained using the Danckwerts boundary conditions. The former are represented by solid curves while the latter are represented by dashed curves. In all the figures, the vertical lines at \( z = \pm L \) represent the reactor boundaries.

Figure 1 shows calculations for the case of no dispersion in the fore section (case 4). The Danckwerts boundary condition is valid at the reactor entrance which is well borne out by the calculations. Initially, the reactor contains reactant at concentration \( \bar{C} \) whereas the aft section contains no reactant. Therefore, the zero gradient boundary condition at the reactor outlet is badly negated for short time.

Results of calculations for the case of no dispersion in the aft section (case 2) are shown in Fig. 2. Initially, the fore and reaction sections contain reactant at concentration \( \bar{C} \). The deviation from the Danckwerts
Table 4. Identification of discrete eigenvalues of $L$

<table>
<thead>
<tr>
<th>$Pe$</th>
<th>$Pe_0$</th>
<th>$Da$</th>
<th>$g$</th>
<th>$M$</th>
<th>$N$</th>
<th>$\lambda$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5.0</td>
<td>21.0</td>
<td>10.0</td>
<td>0.9753961</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$Si=10.0$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$C1=0.04137$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>12.0</td>
<td>21.0</td>
<td>1.0</td>
<td>0.0$^*$</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$Si=20.0$, $S2=6.0$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$C1=0.7295$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>3.0</td>
<td>12.0</td>
<td>1.2</td>
<td>0.23</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$S1=7.25$, $S2=1.25$, $S3=0.6375$, $C2=0.3686$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>4.5</td>
<td>10.5</td>
<td>4.0</td>
<td>0.6</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$S1=2.75$, $S2=4.8375$, $S3=3.1875$, $C2=1.7266$, $C3=2.10168$, $C4=0.6906$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>4.5</td>
<td>10.5</td>
<td>4.0</td>
<td>0.4423369</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$S1=2.75$, $S2=3.9763$, $S3=2.3222$, $C2=1.791$, $C3=1.4366$, $C4=0.6906$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>4.5</td>
<td>10.5</td>
<td>4.0</td>
<td>0.2960305</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$S1=2.75$, $S2=1.128$, $S3=0.3423$, $C2=2.369$, $C3=0.743$, $C5=2.00$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>4.5</td>
<td>10.5</td>
<td>4.0</td>
<td>0.2571429</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$S1=2.75$, $S2=0.7875$, $S3=0.0036$, $C2=2.523$, $C3=1.0052$, $C5=11.523$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>4.0</td>
<td>3.5</td>
<td>5.0</td>
<td>0.4703882</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

$^*$Pe$轴$ or Pe$轴$ = $\infty$

* not defined.

jump condition becomes more pronounced when the dispersion coefficient in the fore section is increased.

Results of calculations for the case where the reactor is fed via a well-stirred tank that precedes the reactor (see case 7 of Table 3) are shown in Fig. 3. The reactor and aft section contain no reactant initially. Results were obtained for two different initial conditions in the stirred tank, the first one corresponding to the situation of no reactant in the tank and the second corresponds to the tank filled with feed. Comparison of the solution obtained using the integral transforms with that obtained using Danckwerts boundary conditions shows the latter appearing below the former for the first initial condition while for the second, the situation is exactly reversed. This is as a result of the
1.2 1.6 2D

Fig. 3. Transient reactor profiles with feed from stirred tank. Effect of initial condition in the stirred tank.

The inertial effect of the tank volume which provides a finite capacitance.

The calculations in Figs. 1–3 did not involve any discrete eigenvalues. Results of calculations for the case where the reactor is succeeded by a stirred tank are shown in Fig. 4. The reactor and the appended sections are filled with reactant at concentration $C_f$ initially. The presence of the reactant (at concentration $C_f$) in the stirred tank results in concentrations which are higher than the ones predicted using Danckwerts boundary conditions in the region near the reactor outlet. The inertial effect of the stirred tank is felt more when the volume of the stirred tank is increased. The deviation of the solution obtained using the integral transforms from the solution obtained using Danckwerts boundary conditions becomes more pronounced as the volume of the post-reactor stirred tank increases. For $q_+ = 0.15$ and 0.333, the dispersion and kinetic parameters lie in category IV and there are no discrete eigenvalues. For $q_+ = 0.8$ and 1.2, the parameters lie in category V and the axial dispersion operator possesses one discrete eigenvalue.

6. CONCLUSIONS

Solutions to the transient problems in axially dispersed systems arising in what were referred to as cases 2, 3, 4 and 7 in Part I[11 have been obtained in this paper. These solutions have been obtained using the properties of the associated axial dispersion operators. These operators possess real continuous and discrete eigenvalues. Expressions for eigenfunctions corresponding to the continuous and discrete eigenvalues, the conditions for existence of discrete eigenvalues and formulae for the number of discrete eigenvalues have been provided. It is essential to identify all discrete
Analysis of axially dispersed systems with general boundary conditions—III

As was pointed out earlier[6, 7], one must be careful in picking the initial condition in an appended section which is assigned infinite length. There is more latitude with the initial condition in the aft section in this respect, since the influence of this section is “washed out” by convective motion. To be on the safer side, when the tube proper is preceded by a fore section infinite in length, the initial reactant/solute concentration should be maintained equal to that in the feed at least beyond the actual entrance region before the tube inlet in the direction of negative infinity.

The nature of the eigenvalues of the axial dispersion operators arising in cases 1–9 (defined in [1]) has been summarized in Table 3. The eigenvalues in cases 5, 6, 8 and 9 are exclusively discrete in nature. In the remaining cases, they may be purely continuous or consist of continuous and discrete counterparts. A complete analysis of axial dispersion models has thus been done using the properties of second order differential operators. The analysis, done essentially for linear rate processes, has been extended to derive stability criteria in adiabatic tubular reactors[8].

Acknowledgement—The authors are grateful to Purdue University for a David Ross grant which made this research possible.

NOTATION

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_+, A_-, A^e_1$</td>
<td>differential expressions defined in eqns (10), (14), (19) and (23) defined in eqn (46)</td>
</tr>
<tr>
<td>$A_+, A^e_1$</td>
<td>axial dispersion operators in cases 2, 3, 4 and 7 defined in eqn (46)</td>
</tr>
<tr>
<td>$D, D_-, D_+$</td>
<td>dispersion coefficients in the reaction section, the fore section and the aft section respectively defined in Table 1 for various cases</td>
</tr>
<tr>
<td>$D_M$</td>
<td>Damkohler number</td>
</tr>
<tr>
<td>$D(L)$</td>
<td>set of functions on which the differential expression $L$ can operate $(L - A_-, A_+, A^e_1$ and $A^e_2$) defined in eqns (12), (17), (21) and (26)</td>
</tr>
<tr>
<td>$dP(\lambda)$</td>
<td>differential projection corresponding to the continuous eigenvalue $\lambda$ of $L$</td>
</tr>
<tr>
<td>$e_j, f_j$</td>
<td>initial dimensionless reactant concentration in the reactor proper and partially mixed appended sections defined in eqn (45)</td>
</tr>
<tr>
<td>$F(x)$</td>
<td>initial dimensionless reactant concentration in the well-mixed aft or fore section</td>
</tr>
<tr>
<td>$G(x, y, \lambda)$</td>
<td>Green's function for $L$ $(L = A_-, A_+, A^e_1$ and $A^e_2$) defined after eqn (64)</td>
</tr>
<tr>
<td>$g$</td>
<td>ratio of volume of well-mixed appended section and half-volume of the reactor defined in Table 1 for various cases</td>
</tr>
<tr>
<td>$h_j$</td>
<td>defined in eqn (47)</td>
</tr>
<tr>
<td>$k_1$</td>
<td>defined in eqn (1.12)</td>
</tr>
<tr>
<td>$L$</td>
<td>half-length of the reactor proper</td>
</tr>
<tr>
<td>$L$</td>
<td>general axial dispersion differential expression</td>
</tr>
<tr>
<td>$M$</td>
<td>defined in eqn (57)</td>
</tr>
<tr>
<td>$m$</td>
<td>greatest lower bound of $L$</td>
</tr>
<tr>
<td>$m_j$</td>
<td>defined in eqn (46)</td>
</tr>
<tr>
<td>$N$</td>
<td>number of discrete eigenvalues of $L$</td>
</tr>
<tr>
<td>$n$</td>
<td>defined in eqn (1.7)</td>
</tr>
<tr>
<td>$Pe_-$, $Pe_+$, $Pe_0$</td>
<td>Peclet numbers in the reaction section, the fore section and the aft section respectively defined in Table 1</td>
</tr>
<tr>
<td>$p(x)$</td>
<td>projection corresponding to the discrete eigenvalue $\lambda$ of $L$</td>
</tr>
<tr>
<td>$p_0$</td>
<td>defined in eqn (58)</td>
</tr>
<tr>
<td>$p_j(x)$</td>
<td>defined in eqns (44) and (49)</td>
</tr>
<tr>
<td>$Q(\omega)$</td>
<td>weight function defined in eqn (3)</td>
</tr>
<tr>
<td>$r_j$</td>
<td>defined in eqn (46)</td>
</tr>
<tr>
<td>$S(\tau)$</td>
<td>reactant source term vector</td>
</tr>
<tr>
<td>$u_j(x)$</td>
<td>eigenfunction corresponding to discrete eigenvalue $\lambda_j$ of $L$—defined in eqns (44) and (49)</td>
</tr>
<tr>
<td>$v_\pm(x, \omega)$</td>
<td>two independent solutions of the equation $L\psi = \lambda \psi$</td>
</tr>
<tr>
<td>$W(x, \omega)$</td>
<td>normalized eigenfunction corresponding to the continuous eigenvalue $\lambda$ of $L$</td>
</tr>
<tr>
<td>$W_j(x)$</td>
<td>normalized eigenfunction corresponding to the discrete eigenvalue $\lambda_j$ of $L$</td>
</tr>
<tr>
<td>$x$</td>
<td>dimensionless axial coordinate</td>
</tr>
<tr>
<td>$y(x)$</td>
<td>defined in eqn (2)</td>
</tr>
<tr>
<td>$z$</td>
<td>dimensional axial coordinate</td>
</tr>
</tbody>
</table>

Greek symbols

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$</td>
<td>defined in eqn (42)</td>
</tr>
<tr>
<td>$\chi$</td>
<td>dimensionless reactant/solute concentration</td>
</tr>
<tr>
<td>$\chi(x)$</td>
<td>dimensionless reactant/solute concentration vector</td>
</tr>
<tr>
<td>$\chi_+(\tau)$</td>
<td>in case 3 and $\chi_-(\tau)$ in case 7</td>
</tr>
<tr>
<td>$\tilde{x}(\omega, \tau)$</td>
<td>transform of $x(\tau)$ w.r.t. the eigenfunction $W(\omega)$</td>
</tr>
<tr>
<td>$\tilde{\chi}(\tau)$</td>
<td>transform of $\chi(\tau)$ w.r.t. the eigenfunction $W_j$</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>continuous eigenvalue of $L$</td>
</tr>
<tr>
<td>$\lambda_j$</td>
<td>jth discrete eigenvalue of $L$ $(j = 1, 2, \ldots, N)$</td>
</tr>
<tr>
<td>$\lambda_0$</td>
<td>defined after eqn (64)</td>
</tr>
<tr>
<td>$\omega$</td>
<td>parameter related to $\lambda$ as $\omega = \frac{\lambda}{\sqrt{\frac{Pe}{Pe_0^2} - 4}}$</td>
</tr>
<tr>
<td>$\omega_1$</td>
<td>defined in eqn (42)</td>
</tr>
<tr>
<td>$\sigma(L)$</td>
<td>set of continuous eigenvalues of $L$</td>
</tr>
<tr>
<td>$\tau$</td>
<td>dimensionless time</td>
</tr>
</tbody>
</table>
**Special Symbols**

- \( \oplus \) direct sum
- \( \mathfrak{C} \) complex number space
- \( \mathcal{D}(\omega) \) defined in eqn (42)
- \( \mathcal{H} \) Hilbert space
- \( L^2 \) space of square integrable functions
- \( \langle \cdot , \cdot \rangle \) inner product in the Hilbert space

**Subscripts**

- \( f \) feed
- well-mixed aft section
- well-mixed fore section

**Superscripts**

- \( * \) complex conjugate

**REFERENCES**


**APPENDIX**

We denote the general axial dispersion operator in cases 2, 3, 4 and 7 as \( L \).

**Theorem 1.** \( L = \{ L, D(L) \} \) is a self-adjoint operator.

**Proof.** For \( L \) for \( \mathcal{A}, \mathcal{A}_- \), we write the following using the inner products defined in eqns (13) and (22).

\[
\langle Lu, v \rangle - \langle u, Lv \rangle = \left[ u, v \right]_{-a} - \left[ u, v \right]_{a}, \forall u, v \in D(L) \quad (1.1)
\]

\[
\left[ u, v \right] = e^{-\omega x} \left\{ u(x) \frac{du^*}{dx} (x) - v^*(x) \frac{dv}{dx} (x) \right\} \quad (1.2)
\]

For \( L = \mathcal{A}^2 \) and \( \mathcal{A}^2_- \), in view of the inner product definitions (18) and (27), we have

\[
\langle \mathcal{A}^2 u, v \rangle - \langle u, \mathcal{A}^2 v \rangle = - \left[ u, v \right]_{-a} \quad \forall u, v \in D(\mathcal{A}^2) \quad (1.3)
\]

\[
\langle \mathcal{A}_-^2 u, v \rangle - \langle u, \mathcal{A}_-^2 v \rangle = \left[ u, v \right]_{-a} \quad \forall u, v \in D(\mathcal{A}_-^2). \quad (1.4)
\]

Since \( u(x), v(x) \in \mathcal{L}_2 \) and \( L u, L v \in \mathcal{H} \), we have

\[
\begin{align*}
\lim_{x \to \pm \infty} w(x) \exp \left( - \frac{Pe}{2} x \right) &= 0 \\
\lim_{x \to \pm \infty} \left( - \frac{dw}{dx} + Pe \frac{dw}{dx} \right) \exp \left( - \frac{Pe}{2} x \right) &= 0
\end{align*}
\]

In each case,

\[
w(x) \exp \left( - \frac{Pe}{2} x \right)
\]

and its first derivative are locally absolutely continuous in the interval \( (-\infty, b) \) or \( (a, \infty) \). Therefore, in view of eqns (1.5),

\[
w'(x) \exp \left( - \frac{Pe}{2} x \right)
\]

is finite at \( x = \pm \infty \). Thus

\[
\lim_{x \to \pm \infty} e^{-\omega x} u(x) \frac{du^*}{dx} (x) = 0 \quad \forall u, v \in D(L) \quad (1.6)
\]

Since the functions belonging to \( D(A) \) and \( D(A_-) \) satisfy homogeneous boundary conditions at \( x = 1 \) and \( x = -1 \) respectively, from eqn (1.6) it follows that the r.h.s. of eqns (1.1), (1.3) and (1.4) are zero. The axial dispersion operators \( L = A, A_-, A_+, A_-^2 \) are thus self-adjoint.

In the following, we state some theorems without providing proofs. The reader is referred to some of our other works[4, 5] for the proofs.

**Theorem 2.** \( L \) is bounded from below with \( m \) being the greatest lower bound of \( L \), \( m \geq n \) where

\[
n = \min \left\{ \frac{Pe^2}{4} + Da, \frac{Pe}{4}, \frac{Pe}{2g} \right\} \quad (1.7)
\]

**Theorem 3.** The spectrum of \( L \) is discrete in the interval

\[
\lambda < \frac{Pe}{4} \quad (1.8)
\]

**Theorem 4.** Condition (40) is the necessary condition for the existence of at least one discrete eigenvalue of \( L \).

**Theorem 5.** Let \( Pe, Pe_0, Da \) and \( g \) be such that conditions (56) are satisfied. Then if eqn (59) is satisfied with \( M \) being defined in eqn (57), then \( L \) has \( M \) discrete eigenvalues. Otherwise, there are \( (M - 1) \) discrete eigenvalues.

**Theorem 6.** Let the parameters \( Pe, Pe_0, Da \) and \( g \) be such that conditions (60) are satisfied. Then \( L \) has \( (M + 1) \) discrete eigenvalues if condition (61) is satisfied. Otherwise, \( L \) has \( M \) discrete eigenvalues.

**Theorem 7.** Let \( Pe, Pe_0, Da \) and \( g \) be such that conditions (62) are satisfied. If conditions (61) and (65) are satisfied, is satisfied when

\[
\lambda_0 = \frac{Pe^2}{4} + Da = \frac{Pe}{4} \quad (1.9)
\]

then \( L \) has one discrete eigenvalue in

\[
\left( \frac{Pe^2}{4} + Da, \frac{Pe}{4} \right) \quad (1.10)
\]

Otherwise, there is no discrete eigenvalue in this interval.

**Theorem 8.** Let \( Pe, Pe_0, Da \) and \( g \) be such that conditions (66) are satisfied. If conditions (65) and (66) are satisfied, then \( L \) has \( M + 1 \) discrete eigenvalues in

\[
\left( \frac{Pe_0^2}{4} + Da, \frac{Pe^2}{4} \right) \quad (1.11)
\]

If either (61) or (65) is not satisfied, then \( L \) has \( M \) discrete eigenvalues in this interval. If both (61) and (66) are violated, then there are \((M - 1)\) discrete eigenvalues in the above interval.

The Green's function for \( L \) is defined as

\[
w(x) = \omega(x) \exp \left( Pe \frac{dx}{2} \right) - \frac{1}{g} \frac{dw}{dx} \exp \left( Pe \frac{dx}{2} \right)
\]
Analysis of axially dispersed systems with general boundary conditions—III

\[ G(x, y, \lambda) = \begin{cases} v_+(y, \omega) \eta_+(x, \omega) / Q(\omega), & x \geq y \\ v_-(y, \omega) \eta_-(x, \omega) / Q(\omega), & x \leq y \end{cases} \] (1.8)

\[ Q(\omega) = \exp \left( - \frac{Pe \omega}{v_+(x, \omega) u_+(x, \omega)} \right) - v_-(x, \omega) u'_-(x, \omega). \] (1.9)

The relation between \( \omega \) and \( \lambda \) is shown in eqn (42). \( v_+(x, \omega) \) and \( u_+(x, \omega) \) satisfy the boundary conditions at \( x = a \) and \( x = b \) respectively.

**Theorem 9.** The continuous eigenvalues of \( L \) lie in

\[ \left[ \frac{PePe_0}{4}, \infty \right). \]

For \( L = A_+ \) and \( A_- \), the differential projection \( dP(\lambda) \) is represented as

\[ dP(\lambda) u = \frac{2Pe_0}{\pi Pe} \frac{d\omega}{d\omega} \int_a^b \Im \{ G(x, y, \lambda) \} u(y) e^{-\frac{Pe \omega}{v_+(x, \omega)} dy. \] (1.10)

For \( L = A^+ \) and \( A^- \), the differential projection \( dP(\lambda) \) has the form

\[ \frac{2Pe_0}{\pi Pe} \Im G(x, y, \lambda) = \frac{W(x, \omega) W(y, \omega)}{\omega}, \]

\[ a < x, y < b. \] (1.13)