CONJUGATED GRAETZ PROBLEMS—II

FLUID–FLUID PROBLEMS

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Abstract—Conjugated Graetz problems [1], involving two co- or counter-currently flowing phases are discussed following the general formalism of [1]. Although the difficulty, which arises from the changing sign of the velocity profile over the total cross-section of the two-fluid conduit of the counter-current problems, may be readily handled by the general formalism, the inclusion of axial heat conduction in the analysis presents special difficulties in obtaining analytical and/or computationally efficient solutions. The present analysis shows that analytical and computationally efficient solutions may be obtained only for these problems where the temperature profile at the entrance of the heating section is known for at least one of the fluids. The solution of a class of problems with long heating sections is obtained utilizing the general formalism together with the Gram–Schmidt orthonormalization process in the spirit of [5]. Problems with low Peclet numbers for both fluids and with an adiabatic section preceding the heating section or problems with very short heating sections are also briefly discussed.

1. INTRODUCTION

The general conjugated Graetz problems have been discussed and analyzed in a preceding paper [1], together with a class of solid–fluid problems which admit simple analytical solutions. Graetz problems with two flowing fluids however present additional difficulties in their solution and are the focus of this paper. Their applications are discussed in some detail in [1, 2], together with an extensive survey of the literature. It was observed in [1] that the treatment of these problems encounters difficulties related to the axial conduction term and to the velocity profile due to the fact that for counter-current flows, it changes sign over the total cross-section of the two-fluid conduit. While the general formalism presented in [1] eliminates the difficulty arising from the velocity profile, the axial heat conduction terms, coupled with either co- or counter-current flows, present formidable difficulties towards analytical and/or computationally efficient solutions. The heat exchanger is presumed to consist of a surface of separation of the fluids that is adiabatic preceding the heat-exchange section, on either side of which the fluids persist in fully developed flow. It will follow from the analysis below, that analytical (and computationally efficient) solutions may be obtained only for those problems where the temperature profile, at the entrance of the heat-exchange section, is known for at least one of the fluids, either for the co- or for the counter-current flow. Problems with low Peclet numbers for both fluids and with adiabatic sections preceding the heating-exchange sections or problems with finite heating sections will be discussed briefly later.

In view of the constraint of an infinite axial domain that is characteristic of our analysis, long heat exchangers best represent the class of problems to be discussed here. Such heat exchangers are characterized by one non-equilibrium end and the other an equilibrium end. We take the non-equilibrium end at \( z = 0 \) and the equilibrium end at \( z = \infty \). The thermally overwhelming fluid, i.e. the fluid with the larger flow heat capacity \( (nC_p) \), enters the exchanger at \( z = \infty \). Consider, thus, the heat transfer problem depicted in Fig. 1. The high Peclet number fluid 1, flowing in the tubular space of the two concentric tubes, is entering the heat-exchange section with a uniform temperature \( T_1 \). The low Peclet number fluid 2 is entering the annular space of the double pipe at \( z = m \). At \( z = m \) and because of the infinitely long heating section the inner fluid temperature also becomes uniformly \( T_1 \). The outer wall is thermally insulated throughout, while the inner wall is insulated for \( z < 0 \). The other geometric and flow characteristics are those of Fig. 1 of [1].

Similarly, we may treat the problem of Fig. 2, where the high Peclet number fluid 2 is entering the heating section in the annular space with uniform temperature \( T_2 \), while the low Peclet number fluid 1 is entering the

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Fig. 2. A counter-current conjugated Graetz problem, with the fluid in the annular space flowing in the positive z direction.

tubular space of the double pipe at \( z = \infty \) with a uniform temperature \( T_2 \). Fluid 2 again attains temperature \( T_2 \) at \( z = \infty \). The outer wall is again thermally insulated, while the inner wall is insulated for \( z < 0 \).

It may be noted from both the above problems that we always take the high Peclet number fluid to enter the heat-exchange section at \( z = 0 \). This allows us to specify its temperature at \( z = 0 \) as being uniformly \( T_1 \), because of insignificant axial heat conduction. Our computations in [2-5] have indicated that we may specify this uniform temperature \( T_1 \), for Peclet numbers higher say than 60 or 70.

The above class of problems is realistic in those cases, where, for example, a liquid–metal fluid is being cooled by a nonmetallic fluid in a tubular heat exchanger which is long enough to ensure the most effective cooling. If the Peclet number of the higher Peclet number fluid 1 is not very small, say \( Pe_1 > 15 \), it is fair to say that the problem may be adequately analyzed by prescribing for fluid 1 the pointwise Danckwerts boundary condition at \( z = 0 \) following the method of [5]. A complete treatment of the problem with two low Peclet number fluids is indeed considerably more involved as we shall see shortly. The analysis and the computations that follow are performed for the problem of Fig. 1. The problem of Fig. 2 and any of the problems with Danckwerts boundary conditions may be analyzed similarly.

2 ANALYSIS

With constant physical properties, neglecting viscous dissipation and assuming axisymmetry, we may write for the problem of Fig. 1, in dimensionless form,

\[
\frac{1}{\eta} \frac{\partial}{\partial \eta} \left( \eta \frac{\partial \theta}{\partial \eta} \right) - \frac{Pe^2}{\eta} \frac{\partial^2 \theta}{\partial \eta^2} + v(\eta) \frac{\partial \theta}{\partial \eta} = 0
\]

(1)

\[
0 < \eta < b - d \\
\eta < b < 1 \\
0 < \eta < \infty
\]

(2)

(3)

where

\[
\frac{\partial \theta}{\partial \eta} (\zeta, 0) = 0
\]

(4)

\[
\frac{\partial \theta}{\partial \eta} (\zeta, 1) = 0
\]

(5)

\[
- \frac{\partial \theta}{\partial \eta} (\zeta, b - d) = B[\theta(\zeta, b - d) - \theta(\zeta, b)]
\]

\[
\frac{\partial \theta}{\partial \eta} (\zeta, b - d) = k \frac{\partial \theta}{\partial \eta} (\zeta, b)
\]

(6)

(7)

(8)

(9)

As we saw in [1], we may define the dimensionless axial energy flow function,

\[
\Sigma(\zeta, \eta) = \int_0^\infty \left[ - \frac{1}{Pe^2} \frac{\partial \theta}{\partial \eta} + v(\eta) \right] 2\eta^\prime \, d\eta^\prime
\]

(10)

\[
0 < \eta < b - d
\]

(11)

such that energy eqn (1) may be represented by the equivalent system,

\[
\frac{\partial \theta}{\partial \eta} = Pe^2 v(\eta) \frac{\partial \theta}{\partial \eta} - Pe^2 \frac{\partial \Sigma}{2 \eta \eta^\prime}
\]

(12)

\[
\frac{\partial \Sigma}{\partial \eta} = 2\eta \frac{\partial \theta}{\partial \eta}.
\]

(13)

From (10) and (11) we obtain, respectively,

\[
\Sigma(\zeta, 0) = 0 \quad \text{all } \zeta
\]

(14)

\[
\gamma \Sigma(\zeta, b - d) = \Sigma(\zeta, b - d) \quad \text{all } \zeta
\]

(15)

while from an energy balance over the whole cross section of the double pipe between \( \zeta \) and \( \zeta + d\zeta \) and integrating we obtain,

\[
\Sigma(\zeta, 1) = \text{constant} - \Sigma(\zeta, 0).
\]

(16)

Since \( (\partial \theta/\partial \zeta) = 0 \) at \( \zeta = \infty \) from (10), (11), (7) and (16) we obtain,

\[
\Sigma(\zeta, 0) = 0 \quad \text{all } \zeta
\]

(17)

which expresses that the dimensionless axial flow in energy is equal and opposite in the two fluids when the outer surface is thermally insulated.

Noting that it may be readily shown[2] that (2) and (5) are respectively implied from (14) and (15), the problem may be written in the matrix form

\[
\frac{\partial}{\partial \eta} F(\zeta, \eta) = LF(\zeta, \eta)
\]

(18)

with

\[
v(\eta) = \begin{bmatrix}
\frac{1}{A} \left( 1 - \frac{\eta^2}{(b - d)^2} \right) & \eta \\
-\frac{1 - \eta^2 + a \ln \eta}{b - \eta^2} & \eta
\end{bmatrix}
\]

(19)

\[
0 < \eta < b - d \quad b < \eta < 1.
\]
and

\[ F(\zeta, \eta) = \begin{bmatrix} \theta(\zeta, \eta) \\ \Sigma(\zeta, \eta) \\ \psi(\zeta, b - d) \end{bmatrix} \]  

subject to boundary conditions (14), (15) and (17). Indeed, boundary condition (3) is readily implied from (17) while (4) is already incorporated into (18).

In view of (14), (15) and (17) we define the inner product between two vectors,

\[ \langle \phi, \psi \rangle = \int_0^{b-d} X(\phi, \psi) \, d\eta + \gamma \int_0^1 X(\phi, \psi) \, d\eta + \frac{\phi_3 \psi_3}{(b - d)B} \]  

(21)

where

\[ X(\phi, \psi) = \frac{4}{\rho \varepsilon} \phi_1(\eta) \psi_1(\eta) + \phi_2(\eta) \psi_2(\eta) \frac{1}{\eta} \]  

(22)

then \( L \) is symmetric in the Hilbert space \( \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H} \) with the domain,

\[ D(L) = \{ \phi \in \mathcal{H} : L \phi \text{exists and} \} \]  

(23)

as shown in [2]. Hilbert spaces \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) have been defined in [1]. Thus, we obtain the eigenvalue problem,

\[ L \phi_j = \lambda_j \phi_j \]  

(24)

The properties of \( L \) are analogous to those of the operator \( L \) discussed in [1]. For the co-current problems operator \( L \) does not possess \( \lambda_0 = 0 \) as an eigenvalue; this may be proved in a manner exactly analogous to the corresponding proof in [4]. For the counter-current problems we may similarly show that \( \lambda_0 = 0 \) is not an eigenvalue, except for the very special case where the following equality holds:

\[ \int_0^{b-d} \eta \psi_1(\eta) \, d\eta + \int_0^1 \eta \psi_2(\eta) \, d\eta = 0. \]  

(25)

The usual expansion theorem is required for the solution of the problem that we shall discuss next:

\[ t = \sum_{j=1}^\infty \left[ \frac{\langle t, \phi_j \rangle}{\| \phi_j \|^2} \phi_j + \frac{\langle \phi_j, \phi_j \rangle}{\| \phi_j \|^2} \phi_j \right] = \sum_{j=1}^\infty \left[ \frac{\langle t, \phi_j \rangle}{\| \phi_j \|^2} \phi_j \right]. \]  

(26)

We seek the solution in the form of the expansion (26); to that effect we want to determine the inner products of the solution vector with the eigenvectors. Since the solution vector, \( F \), for every \( \zeta \) belongs to the domain of \( L \), from (18) and (24) we have,

\[ \frac{d}{d\zeta} \langle F, \phi_j \rangle = \lambda_j \langle F, \phi_j \rangle \]  

(27)

from which we obtain

\[ \langle F, \phi_j \rangle = C_j e^{s \zeta}. \]  

(28)

Since the solution is bounded at \( \zeta = \infty \), we have

\[ C_j = 0. \]  

(29)

while from (28) for \( \zeta = 0 \), we obtain

\[ C_j = \langle F(0, \eta), \phi_j \rangle = A_j \| \phi_j \|^2. \]  

(30)

Then, we may write the solution as,

\[ F = \sum_{j=1}^\infty A_j e^{s \zeta} \phi_j(\eta). \]  

(31)

If we now write from (31) the solution for the first components and for \( 0 < \eta < b - d \), we have,

\[ \theta(\zeta, \eta) = \sum_{j=1}^\infty A_j e^{s \zeta} \phi_j(\eta) = \sum_{j=1}^\infty A_j e^{s \zeta} x_j(\eta) \]  

(32)

In (32) \( x_j(\eta) \) is defined to be the portion of \( \phi_j(\eta) \) for \( 0 < \eta < b - d \). From (32) and (6) we may obtain,

\[ \theta(0, \eta) = \sum_{j=1}^\infty A_j x_j(\eta) = 1. \]  

(33)

The linear independence of the set \( \{ x_j(\eta) \} \) derives from the linear independence of the set \( \{ \phi_j(\eta) \} \). Hence, the solution of problem may be reduced to the generation of an orthonormal set from \( \{ x_j(\eta) \} \) and thus to the computation of the expansion coefficients of \( 1 \) in (33). It may be noted that the procedure is very analogous to the procedure we used in [2, 5]. The natural inner product to use for the orthonormalization process is

\[ \langle \phi(\eta), \psi(\eta) \rangle = \int_0^{b-d} \eta \psi_1(\eta) \, d\eta. \]  

(34)

As we have already seen in [2, 5], it is a simple computational procedure to determine constants \( \{ c_{k,j}, j, k = 1, 2, 3, \ldots \} \) such that

\[ \psi(\eta) = \sum_{j=1}^\infty c_{k,j} x_j(\eta) \]  

(35)

where \( \{ \psi(\eta) \} \) is the orthonormal set generated from the set \( \{ x_j(\eta) \} \). Expanding unity in terms of this newly generated set, we obtain,

\[ 1 = \sum_{j=1}^\infty \int_0^{b-d} \eta \psi_j(\eta) \, d\eta = \sum_{j=1}^\infty B_j \sum_{k=1}^\infty c_{k,j} x_k(\eta), \]  

(36)
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Thus, by virtue of (33), from (36) we have,

\[ A_k = \sum_{j=1}^{\infty} B_j \epsilon_{jk} \]  

(38)

The evaluation of the set \( \{ \epsilon_{jk} \} \) has been described in detail in [2, 5]. Of course, here we use the generating set \( \{ x_k(\eta) \} \) with the inner product (34). For the evaluation of the set \( \{ \epsilon_{jk} \} \) the following integrals are required,

\[ 0_k = \int_0^1 x_j x_k \eta \, d\eta \]  

(39)

the computation of which is discussed in [2]. Further, from (37) we may write, in view of (35),

\[ B_j = \sum_{k=1}^{\infty} \epsilon_{jk} \int_0^1 x_k(\eta) \eta \, d\eta \]  

(40)

where the computation of the integrals appearing in (40) is discussed in [2].

Definitions, expressions and other computational details for the various Nusselt numbers, bulk temperatures and asymptotic Nusselt numbers may be found in [2].

3. COMPUTATIONAL ASPECTS

The integration of the eigenvalue problem (24) is discussed in detail in [2]. Thus, for laminar flows and for the tubular region the integration is achieved using Kummer's functions [2-5], while for the annular region a series type solution is employed. This integration technique is indeed most efficient [2]. For example, the computation of 14 eigenvalues with an accuracy of 10 significant figures requires 6-7 CP sec on CDC 6600 computer, depending upon the Peclet number and the other constants of the problem. The expansion coefficients \( A_j \) are computed according to eqn (38). The computation of the set \( \{ \epsilon_{jk} \} \) has been discussed in [2, 5]; we saw, for example, that the computational effort increases much faster than the number of terms included in the computations.

There appear to be two sources of error in the computation of the expansion coefficients. One source of error is the effect of the higher eigenfunctions on the expansion of any quantity and thus on the computation of the \( A_j \)'s in view of eqns (36)-(38). Minimization of this error would call for the usage of a large number of terms. The other source of error is the Gram-Schmidt procedure which tends to be unstable as more and more terms are used [6, 7]. The optimal number of terms is determined such that the expansion coefficients \( A_j \) and also, more critically, the final temperature computations, change the least by increasing or decreasing \( j \). For the computations reported next, we found that 8-12 terms provide optimal results with an accuracy of better than 0.1%, except at the very beginning of the thermal entrance region. Further, the computation of 8-12 expansion coefficients required only 3-7 CP 6600 sec.

Finally, the computation of 168 temperature points, with 8-12 series terms, typically used for Figs. 3-5 required 3-4 CP 6600 sec. Further computational details may be found in [2].

4. RESULTS AND DISCUSSION

Computations have been performed for the conjugated problem that was analyzed in Section 2 with the follow-

Fig. 3. Radial profiles of dimensionless temperature for various axial distances for the problem of Fig. 1. \( Pe = 20, Pe_1 = 70, b = 0.6, d = 0.05, k = 35 \) and \( B = 600 \).

Fig. 4. Radial profiles of dimensionless temperature for various axial distances for the problem of Fig. 1. \( Pe = 40, Pe_1 = 70, b = 0.6, d = 0.05, k = 35 \) and \( B = 600 \).
Fig. 5. Radial profiles of dimensionless temperature for various axial distances for the problem of Fig. 1. $P_e = 70$, $P_e_1 = 70$, $b = 0.6$, $d = 0.05$, $k = 35$ and $B = 600$.

We found that the expansion coefficients change considerably with the number of terms included in the Gram-Schmidt process. Fortunately, the effect on the final temperature computations is minimal however, except very close to $\xi = 0$ for the smaller Peclet number range. Further, we found that 8-12 terms provide optimal results for the range of Peclet numbers studied as it was also discussed in the preceding section. Ten or less terms give better results for the lower Peclet number range, while more than 15 terms make the Gram-Schmidt process very unstable for all Peclet numbers. Difficulties may still exist in computing temperature profiles for small $\xi$'s with small Peclet numbers. The causes for these difficulties and their remedies will be discussed later, after presenting and discussing the temperature profiles.

Radial temperature profiles for various axial distances are shown in Figs. 3-5 for $P_e_2 = 20$, 40 and 70, respectively. The most striking characteristic of these profiles is that they are virtually uniform in the annular space which is because of the high $k = (k_2/k_1)$ value of 35. That is, since $k_2$ is so much larger than $k_1$, heat is immediately distributed across the annular space with minimal radial temperature gradients. Also, in view of eqn (5), the temperature gradient, on the common wall, in the annular space need only be very small compared to the gradient in the tubular region, since $k$ is so large. It is implied from the above that, when the temperature gradient, on the common wall, in the tubular space tends to zero (i.e. for $\xi \to 0$ and $\xi \to \infty$) the temperature profile in the annular space becomes entirely uniform, which was anticipated for $\xi \to 0$ but it was not as obvious to us for $\xi \to 0$. However, the implications of the high $k$ value go even further; because of the flow direction, the axial temperature gradient at $\xi = 0$ also becomes zero. Indeed, because of the high Peclet number in the tubular space, $\frac{\partial \theta}{\partial \xi} |_{\xi=0} = 0$ with $0 < \eta < b - d$. For the annular space if we assume that $\frac{\partial \theta}{\partial \xi} |_{\xi=0} \neq 0$, then obviously this gradient can only be negative. Let $\theta_{20}$ be the uniform temperature at $\xi = 0$ in the annular space. An overall energy balance, for the whole cross section of the double tube between $\xi = 0$ and $\infty$ will give, in dimensionless form,

$$\int_{0}^{b-d} A v_1(\eta) \eta d\eta = \frac{\gamma}{P_e_2 \cdot P_e_1} \int_{0}^{1} \frac{\partial \eta}{\partial \xi} |_{\xi=0} \eta d\eta$$

$$+ A y \theta_{20} \int_{0}^{1} |v_2(\eta)| \eta d\eta.$$  (41)

Now let $\theta_{20}$ be the uniform temperature in the annular space for $\xi = -\infty$ (see Fig. 1). Obviously $\frac{\partial \theta}{\partial \xi} |_{\xi=-\infty} = 0$ for all $\eta$ and an overall energy balance between $\xi = -\infty$ and $\infty$ would give, in dimensionless form,

$$\int_{0}^{b-d} A v_1(\eta) \eta d\eta = A y \theta_{20} \int_{0}^{1} |v_2(\eta)| \eta d\eta.$$  (42)

If $\frac{\partial \theta}{\partial \xi} |_{\xi=-\infty} < 0$, it would be implied from eqns (41) and (42) that

$$\theta_{20} < \theta_{20}$$  (43)

which is impossible, since it would be required that heat be conducted from a lower to a higher temperature between $\xi = -\infty$ and 0. Thus,

$$\frac{\partial \theta}{\partial \xi} |_{\xi=0} = 0$$  (44)

and in view of (41) and (42).

$$\theta_{20} = \theta_{20}$$  (45)

where $\theta_{20}$ may be readily computed from (42). It would be of interest to examine whether the above arguments could be generalized to prove (44) for the case that the temperature in the annulus at $\xi = 0$ is not uniform. Note that the above arguments and hence eqns (44) and (45) hold exactly true if and only if the annular temperature is uniformly $\theta_{20}$ at $\xi = 0$. Although the foregoing condition is mathematically correct only for $k \to \infty$, it appears to be practically true for $k = 35$. For smaller $k$ values non-uniform temperature profiles in the annulus should be expected. Note also, in passing, that in the same approximate sense as above, eqn (42) provides also the smallest $P_e_2$ for which this particular problem has solution; this Peclet number is found to be $P_e_2 = 8.82$, for which $\theta_{20} = 1$. The computational difficulty in calculating temperature profiles for small distances increases as we approach this $P_e_2$ limit from above.

We are now ready to undertake the discussion, previously deferred, on the difficulty of computing the
expansion coefficients. It may be observed, that in all problems studied earlier[2-5] zero axial temperature gradients at \( \zeta = 0 \) have been *exclusively* associated with high Peclet numbers. The present problem, however, demands a zero axial temperature gradient at \( \zeta = 0 \) when the Peclet number is not high. This very fact puts quite a strain on the set of the eigenfunctions to produce an expansion for the temperature at \( \zeta = 0 \) that will show a zero axial gradient. It is exactly close to \( \zeta = 0 \) with small Peclet numbers that the derived solution fails to produce correct results. An additional strain imposed on the set of the eigenfunctions is the demand to produce *almost* flat temperature profiles in the annular space for all \( \zeta \) values. It is understandable then, that many expansion terms would be required for all the above demands to be met and that the computation of the expansion coefficients, using the Gram–Schmidt process, would be very sensitive to the number of terms used. Unfortunately the Gram–Schmidt process tends to become unstable when many terms are used. It should be noted, however, that the computation of the expansion coefficients is such that the expansion of 1 (see eqn 33) is *actually* achieved always in the tubular space, irrespective of the number of terms used, for \( j \) between 8 and 16.

It is the temperature in the annular space at \( \zeta = 0 \) that the expansion fails to produce correctly. The higher the \( P_e \), the smaller the values of \( \zeta \) for which the correct temperature profiles may be obtained. It is clear then that if the solution was producing the correct numbers at \( \zeta = 0 \), it would produce correct results for the whole tube.

In the light of the above discussion, for small axial distances with small Peclet numbers \((8.82 < P_e < 15)\) the following modified method of solution may be followed. The annular uniform temperature at \( \zeta = 0 \) is first computed from eqn (42) and then the Gram–Schmidt procedure, as described in Section 2, is applied using the whole radial domain, \([0 < \eta < b - d, b < \eta < 1]\). Instead of only the tubular part of the domain. This way the right expansion is produced at \( \zeta = 0 \) and thus the solution ought to produce the correct results for the whole axial domain. This modified technique for the computation of the expansion coefficients would require more computation time, however.

Finally, the uniform temperatures at \( \zeta = 0 \) in the annular space, computed according to eqn (42), for \( P_e = 20, 40 \) and 70 are 0.441, 0.221 and 0.126, while the actual dimensionless lengths of the heating sections, as computed by the solution, are (approximately) 1.6, 0.5 and 0.2, respectively.

It is expected that the present solution methodology will not encounter difficulties with similar problems under less stringent conditions. Indeed, the physical parameters of the present choice are, possibly, overly stringent. It seems to us that the computational efficiency with which the solution affords numbers is commensurate with its simplicity derived from explicit expansion coefficients.

The second problem of Fig. 2 may be treated in a very similar manner, where the Gram–Schmidt process will be applied on the eigenfunctions in the annular space. For the problems where none of the Peclet numbers is sufficiently large, the use of the pointwise Danckwerts boundary condition, at the entrance of the larger Peclet number fluid, ought to provide a good description of the physical problem[2, 5]. This case may be also treated using the Gram–Schmidt process[2, 5]. If both Peclet numbers are lower than, say 15, then the whole axial domain ought to be used in conjunction with the method discussed in [2], Chap. 6. Thus, a set of algebraic equations will result for the computation of the expansion coefficients. This same method may also be applied for the solution of conjugated problems with finite heating sections. In view of the complexity of this latter solution however, it is doubtful that the effort could be justified, since, after all, the actual heating sections, computed with seminfinite positive axial domains, are relatively short as the computations in this paper and in [2, 3, 5] have shown.

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**NOTATION**

- **A** a constant \((= P_e_2/P_e_1)\)
- **A_j** expansion coefficient corresponding to \( \lambda_j \); eqns (30) and (31)
- **a** a constant, eqn (9) (see also [1])
- **B_0** Biot number for the inner wall of the double tube \((= hR/k_i)\)
- **b** a constant, eqn (37)
- **C_i, C'_i, C''_i** integration constants corresponding to \( \lambda_i \), \( \lambda_i^+ \) and \( \lambda_i^- \), respectively, eqns (28)-(30)
- **c_p** specific heats; \( i = 1 \) for the tubular fluid and \( i = 2 \) for the annular fluid
- **D** domain of an operator
- **d** dimensionless width of the inner wall of the double tube
- **F** the solution vector in \( \mathcal{H} \)
- **f** any three component vector in \( \mathcal{H} \)
- **H** Hilbert spaces
- **h** heat transfer coefficient for the resistance of the inner wall \((= k_i ((r, ln (b/(b - d))))\)
- **k** \((= k_1/k_i (b/(b - d)))\)
- **k_{1,2}** conductivities of fluids 1 and 2, respectively
- **k_i** conductivity of the inner wall
- **L** the linear differential operator of eqn (19)
- **O_1** the integral of eqn (39)
- **P_e** Peclet number \((= P_e_2)\)
- **P_e_1** Peclet number of fluid \( i = V_i R_p c_p (k_i) \); \( i = 1 \) for the tubular fluid and \( i = 2 \) for the annular fluid
- **R** the radius of the outer tube of the double pipe
- **R** the set of real numbers
- **r** radial variable
- **R_i** radius of the inner tube of the double tube
- **T** temperature
T₁, T₂  characteristic temperatures, Figs. 1 and 2
v  dimensionless velocity profile, eqns (8) and (9)
v₁, v₂  dimensionless v₁1 and v₁2, respectively, eqns (8) and (9)
v₂₁, v₂₂  velocities of fluids 1 and 2, respectively
X  the function of φ and ψ defined by eqn (22)
x₁  the function of η defined by eqn (32)
z  axial variable

Greek symbols
γ  a constant ( = k₂/k₁)
ε  a constant, eqn (35)
ζ  dimensionless z (= z/Rₚₑ)
η  dimensionless r ( = r/R)
θ  dimensionless T ( = (T₁ - T₂)/(T₁ - T₃))
θ₀  a dimensionless temperature defined above eqn (41)
θₖ  a dimensionless temperature defined below eqn (41)
λ₁, λ⁺₁, λ⁻₁  eigenvalues, positive and negative eigenvalues, respectively
ρ₁, ρ₂  densities of fluids 1 and 2, respectively
Σ  dimensionless flow function, eqns (10) and (11)
φ  any function of η in X₁
ψ  any function of η defined by eqn (35)
ψ₀  a function of η defined by eqn (35)
(,)  the inner product of eqn (21)
‖‖  the norm that corresponds to (,)