Limited Feedback Precoding for Orthogonal Space-Time Block Codes

David J. Love
School of Electrical and Computer Engineering
Purdue University
West Lafayette, IN 47907
djlove@ecn.purdue.edu

Robert W. Heath, Jr.
Dept. of Electrical and Computer Engineering
The University of Texas at Austin
Austin, TX 78712
rheath@ece.utexas.edu

Abstract—Orthogonal space-time block codes (OSTBCs) are an efficient solution for obtaining full diversity order in Rayleigh fading channels. Unfortunately, OSTBCs exist for only certain numbers of transmit antennas and do not provide array gain like diversity techniques that exploit transmit channel information. When channel state information is available at the transmitter, linearly precoded space-time codes can be used to support different numbers of transmit antennas and to improve array gain. Precoding generally requires complete channel knowledge thus motivating limited feedback methods such as channel quantization or antenna subset selection. This paper investigates limited feedback precoding that uses a codebook of matrices known a priori to both the transmitter and receiver. Using the codebook, the receiver chooses a matrix based on current channel conditions and conveys the optimal codebook matrix to the transmitter over an error-free, zero-delay feedback channel. A precoder matrix selection criterion is proposed that relates directly to minimizing the probability of symbol error of the precoded system. Low average distortion codebooks based on Grassmannian subspace packing are derived for the optimal codeword selection criterion. It is shown that codebooks designed by this method provide full diversity order in Rayleigh fading channels.

I. INTRODUCTION

Spatial diversity, obtained through the use of sufficiently spaced antennas, is one method for combatting the detrimental effects of fading in wireless systems. In particular, transmit diversity is attractive because the antennas can be located at the base station and the benefits shared by all users. Obtaining the benefits of multiple transmit antennas without the use of channel knowledge at the transmitter requires the use of special space-time signaling schemes such as space-time codes (see for example [1]–[3]). Orthogonal space-time block codes (OSTBCs) are practical space-time codes due to the simplicity of encoding and optimal decoding [2], [3]. OSTBCs exist only for certain numbers of transmit antennas, however, and this limits their application. Further, OSTBCs suffer from a significant array gain penalty compared to closed-loop techniques such as beamforming. Applying “closed-loop fixes” to OSTBCs can solve both of these problems and has been the subject of recent work on closed-loop space-time block coding [4]–[11].

A convenient way to unify the presentation in these papers is through the construct of linear precoding. With linear precoding, a linear transformation is applied to the spatio-temporal codeword before transmission. By restricting the linear transformation, it is possible to include other closed-loop methods such as transmit antenna selection (see for example [7], [8]). Linear precoding methods can be classified based on the type of channel information available at the transmitter: the instantaneous channel, statistics of the channel, and partial or quantized channel state information. Partial or quantized channel state information [5]–[8] is the most practical of the three methods for frequency division duplexing (FDD) wireless systems. Because the forward and reverse channels fade independently, channel feedback can be sent from the receiver to transmitter. This paper develops limited feedback techniques for OSTBCs. Previous work on limited feedback space-time coding has focused primarily on the case of antenna subset selection [7], [8]. Performance can be improved, however, by letting the linear precoding matrix to have a more general form [5], [6]. Limited feedback precoders were explored in [5] using precoders designed with quantized versions of the forward-link channel. Quantizing the channel directly, however, is a computationally complex problem that often yields too much feedback for a given performance target. For this reason, it is of interest to use limited feedback techniques that deal directly with the broader problem of quantizing the set of precoding matrices.

In this paper, we propose a new form of limited feedback precoding using the general codebook framework for limited feedback precoder selection as in [6] and limited feedback covariance selection as in [12], [13]. In this model, the receiver chooses the linear precoding matrix from a finite cardinality codebook of possible precoding matrices that is designed off-line and available to both the transmitter and receiver. The chosen matrix index is then conveyed back to the transmitter uses a limited number of feedback bits. We present a method for choosing the optimal precoding matrix from the codebook and a criterion for codebook designing. Because we use a finite codebook, the optimal precoding matrix is simply selected by performing a brute force search over all possible codebook precoder matrices. We show that the precoder matrix codebook should be thought of as a set of subspaces. The design of this finite set of subspaces is shown.

Robert Heath was supported in part by the Texas Advanced Research (Technology) Program under Grant No. 003658-0614-2001.
to relate to packing subspaces in the Grassmann manifold using the chordal distance metric. Furthermore, we prove that these design methods yield limited feedback precoders that obtain a diversity order equal to the product of the number of transmit and receive antennas.

Section II provides a general overview of the linear precoding system under consideration. Section III derives a precoder selection criterion for limited feedback OSTBC precoding. A codebook design criterion is developed in Section IV. It is shown in Section V that the designed codebooks provide full diversity order in Rayleigh fading channels. Monte Carlo simulation results are presented in Section VI. We conclude in Section VII.

II. SYSTEM OVERVIEW

The paper will consider limited feedback precoding for OSTBCs transmitted on $M_t$ transmit antenna and received on $M_r$ receive antenna. An OSTBC encoder produces an $M \times T$ (with $M < M_t$) space-time code matrix $C(k) = [c_1(k) \ c_2(k) \ldots \ c_r(k)]$ at the $k^{th}$ transmission from a block of $n_s$ symbols $s_1, s_2, \ldots, s_{n_s}$ taken from a constellation $S$. We assume the power is constrained such that [14]

$$C(k)C(k)^* = \left( \sum_{l=1}^{n_s} |s_l|^2 \right) I_M. \quad (1)$$

We will assume that $E_{s_l}[|s_l|^2] = 1$.

Before transmission, the space-time codeword is premultiplied by an $M_r \times M$ matrix determined by the function $F[\cdot]$. We define the function as the mapping $F : \mathbb{C}^{M_r \times M} \rightarrow \mathcal{F} = \{F_1, F_2, \ldots, F_N\}$. Because the precoder is restricted to lie in the codebook $\mathcal{F}$, the receiver can convey the chosen matrix, $F$, to the transmitter and receiver. We will impose a peak power limit for all $F' \in \mathcal{F}$ by requiring that $\lambda_1(F') \leq 1$ for all $F' \in \mathcal{F}$. We will further restrict the codebook matrices in $\mathcal{F}$, but this restriction will be derived in Section III.

We will model the channel between the transmitter and receiver as a spatially uncorrelated, memoryless linear channel that is constant over several codeword transmissions before independently taking on a new value. Assuming perfect sampling and synchronization, the the system input-output expression is given by

$$Y = \sqrt{\frac{\rho}{M}}HF[H]C + W \quad (2)$$

where $H$ is the $M_r \times M_t$ channel matrix with independent entries distributed as $CN(0,1)$, $W$ is an $M_r \times T$ noise matrix with independent entries distributed according to $CN(0,1)$, and $\rho$ is the signal-to-noise ratio (SNR). $F[H]$ corresponds to the evaluation of the mapping function $H$ for the current channel realization, producing an $M_r \times M_t$ precoding matrix. The receiver decodes $Y$ using optimal maximum likelihood (ML) decoding the effective channel $HF[H]$.

To implement the system described by (2), two main problems must be solved. First, a method for selecting of an optimal precoder given the channel realization from a codebook of matrices $\mathcal{F}$ must be derived. Second, a codebook design criterion must be developed for $\mathcal{F}$ given the chosen selection criterion. Solving these problems is the objective of this paper.

III. SELECTION CRITERION

We first solve the problem of deriving a selection function $F[\cdot]$. The following analysis assumes an arbitrary set $\mathcal{F} \subset \mathcal{L}(M_t, M)$ and ML decoding.

Our goal in this paper is to minimize the symbol error rate (SER) given $H$ denoted by $Pr(\text{ERROR} \mid H)$. Closed-form expressions for the SER would be extremely difficult to obtain because they would be a function of the effective channel $HF[H]$ that is in general no longer matrix Rayleigh fading.

Using the ML detection properties of OSTBCs, it can be shown that

$$Pr(\text{ERROR} \mid H) \leq \exp \left( -\gamma \|HF[H]\|_F^2 \right) \quad (3)$$

where $\gamma$ is a function that depends on $M$, $\rho$, and $S$ [6, 14]. Because $\gamma$ is fixed, minimizing the SER bound in (3) requires the maximization of $\|HF[H]\|_F$. Therefore, we can state the following criterion for precoder selection:

Selection Criterion: Choose the linear precoder from the codebook $\mathcal{F}$ according to

$$F[H] = \arg \max_{F' \in \mathcal{F}} \|HF'[\|_F. \quad (4)$$

This selection criterion can be easily implemented by performing a matrix multiplication and computing a Frobenius norm for each of the $N$ codebook matrices. Ties in effective channel norms are broken arbitrarily by selecting the precoding matrix with the lowest index because they occur with zero probability.

To bound the performance with quantized precoding, we will characterize an optimal unquantized precoder over the set $\mathcal{L}(M_t, M)$. Note that an optimal precoding matrix $F_{\text{opt}}$ will not be unique over $\mathcal{L}(M_t, M)$ because for any $U \in \mathcal{U}(M_t, M)$, $\|HF_{\text{opt}}\|_F = \|HF_{\text{opt}}U\|_F$. The singular value decomposition (SVD) of $H$ is given by

$$H = V_L \Sigma V_R^* \quad (5)$$

where $V_L \in \mathcal{U}(M_r, M_r)$, $V_R \in \mathcal{U}(M_t, M_t)$, and $\Sigma$ is an $M_r \times M_t$ diagonal matrix with $\lambda_j(H)$ at entry $(j,j)$. Let the matrix formed from the first $M$ columns of $V_R$ be denoted...
by $\nabla_R$. The following lemma, proven in [15], [16], shows an optimal matrix in this unquantized scenario.

Lemma 1: An optimal unquantized precoder $F_{\text{opt}}$ over $\mathcal{L}(M_t, M)$ that maximizes $\|HF_{\text{opt}}\|_F^2$ is given by $F_{\text{opt}} = \nabla_R$.

Lemma 1 is striking because it says that not only should $\lambda_1\{F_{\text{opt}}\} = 1$ but $\lambda_j\{F_{\text{opt}}\} = 1$ for $1 \leq j \leq M$. Intuitively, this means that we should always transmit at full power on the precoder effective channel modes when performing optimal precoding and thus perform no waterfilling. $\nabla_R$, however, is not known at the transmitter, and a limited feedback path must be used to convey a sub-optimal precoder. The following lemma shows that linear precoders should always have unit singular values under a maximum singular value constraint.

Lemma 2: If $F' \in \mathcal{L}(M_t, M)$ has a singular value decomposition $F' = U_L\Lambda U_R^*$ and $\lambda_M\{F'\} < 1$, then $F = U_L[I_M 0]^T$ satisfies $\|HF'\|_F < \|HF\|_F$.

Proof: Assume $F'$ has a singular value decomposition as shown in the first lemma. Then

$$\|HF'\|_F = \|HU_L\Gamma\|_F < \|HU_L[I_M 0]^T\|_F = \|HF\|_F,$$

using the invariance of the Frobenius norm to unitary transformation, the singular value bound for $F'$, and $F = U_L[I_M 0]^T$.

The result in Lemma 2 reveals that the precoder should always be designed to have singular values that are as large as possible. Because of the maximum singular value restriction, the precoder should be chosen from the set $\mathcal{U}(M_t, M)$ rather than $\mathcal{L}(M_t, M)$. For this reason we will restrict $F \in \mathcal{U}(M_t, M)$ and thus only design our precoders over $\mathcal{U}(M_t, M)$.

IV. CHORDAL DISTANCE PRECODING

In this section, we derive a codebook design criterion that uses the codebook selection function $F[\cdot]$ derived in Section III. To derive the design criterion, we will define and minimize a distortion function that relates to the average SER. The codebook design criterion turns out to use the famous applied mathematics problem of Grassmannian subspace packing.

A. Distortion Function

Just as in the vector quantization (VQ) literature, we will propose a distortion measure that is a function of the channel and then find a codebook that minimizes a bound on the average distortion. This distortion function must differ, however, from the distortion functions commonly used in VQ such as mean squared error because we are interested in improving the average system performance rather than improving the quality of the reconstructed precoder at the transmitter.

Given the “best” precoding matrix in the codebook given a channel realization $F[H] \in \mathcal{F}$, we would like the quantized equivalent channel $HF$ to provide SER performance close to that provided by the optimal precoded equivalent channel $HF_{\text{opt}}$. Consider the total effective power $\|HF[H]\|_F^2$, which according to (3) relates to the SER. We propose to design the codebook by considering, as a measure of distortion, the average of the loss in received channel power.

To find a good codebook for many channel realizations, we are interested in the average distortion of our codebook given by

$$E_{\mathcal{F}} \left[ \min_{F' \in \mathcal{F}} (\|HF_{\text{opt}}\|_F^2 - \|HF'\|_F^2) \right].$$

It was shown in [15], [16] that

$$\leq E_{\mathcal{F}} \left[ \lambda_1^2 \{H \} \right] E_{\mathcal{F}} \left[ \min_{F' \in \mathcal{F}} \frac{1}{2} \|\nabla_R \nabla_R - F'F'\|_F^2 \right].$$

Thus by bounding the distortion function, we can think of the limited feedback performance as being characterized by two different terms. The first term relates to the distribution of the maximum channel singular value, while the second term represents the “quality” of the codebook $F$.

B. Codebook Design Criterion

To propose a design criterion for $\mathcal{F}$, we must first review some properties and notations dealing with finite subsets of $\mathcal{U}(M_t, M)$. The column space of each matrix in $\mathcal{U}(M_t, M)$ generates an $M$-dimensional subspace of $\mathbb{C}^M$. The set of all possible column spaces of matrices in $\mathcal{U}(M_t, M)$ is the complex Grassmannian manifold $\mathcal{G}(M_t, M)$ [17]. The chordal distance on the Grassmannian manifold is defined as the distance between subspaces $\mathcal{P}_{F_1}$ and $\mathcal{P}_{F_2}$, given by

$$d(F_1, F_2) = \frac{1}{\sqrt{2}} \|F_1F_1^* - F_2F_2^*\|_F.$$  

Let $\mathcal{V} = \{\mathcal{P}_{F_1}, \mathcal{P}_{F_2}, \ldots, \mathcal{P}_{F_N}\}$ be a set of subspaces generated by the codebook matrices $\mathcal{F} = \{F_1, F_2, \ldots, F_N\}$. This set of subspaces $\mathcal{V} \subset \mathcal{G}(M_t, M)$ is a packing of subspaces in $\mathcal{G}(M_t, M)$. Just as in binary coding theory, a packing can be described by its minimum distance

$$\delta = \min_{1 \leq k < l \leq N} d(F_k, F_l).$$

The Grassmannian subspace packing problem is the problem of choosing the set of $N$ subspaces in $\mathcal{G}(M_t, M)$ such that $\delta$ is as large as possible.

It was shown in [15], [16] that the “codebook quality” term in (7) can be approximately bounded as

$$E_{\mathcal{F}} \left[ \min_{F' \in \mathcal{F}} \frac{1}{2} \|\nabla_R \nabla_R - F'F'\|_F^2 \right]\leq M + N \left( \frac{\delta}{2\sqrt{M}} \right)^{2M_tM + o(M_t)} \left( \frac{1}{4} \delta^2 - M \right).$$

Differentiating (10) and using the fact that $\delta < \sqrt{M}$ shows that the bound is a decreasing function of $\delta$ when $2M_tM + o(M_t) > 2/3$. The probabilistic analysis in [18] shows that this is always satisfied for any $M_t$ when $M = 1$, and it is
also clear that this assumption is satisfied for large $M_r$. Thus (10) is approximately minimized by maximizing $\delta$. We have now established that designing low distortion codebooks is equivalent to packing subspaces in the Grassmann manifold using the chordal distance metric.

V. PERFORMANCE ANALYSIS

Because obtaining a closed-form SER expression is virtually impossible to derive in closed-form, we will characterize the diversity of our limited feedback precoders. A signaling scheme obtains diversity of order $d$ if

$$
d = -\lim_{\rho \to \infty} \frac{\log (P_t(\text{ERROR}))}{\log(\rho)}.
$$

(11)
The asymptotic performance of limited feedback precoding can be bounded using the SER result in (3). Thus in order to understand the diversity performance of limited feedback precoding, we will bound the channel gain $\max_{F' \in \mathcal{F}} \|HF'\|_F^2$.

Using the Poincaré separation theorem,

$$
\max_{F' \in \mathcal{F}} \|HF'\|_F^2 \leq \|H\|_F^2.
$$

Note that $\|H\|_F$ is the post-processing channel gain of an $M_t$ antenna OSTBC that is known to give a diversity of order $M_t/M_r$ [3]. Therefore, the diversity order of our limited feedback precoders is less than or equal to $M_t/M_r$. Now all that is needed is a lower bound on diversity order.

Let $i^{th}$ column of codebook matrix $F_k$ be denoted by $f_{k,i}$. The following lemma will be needed to lower bound $\max_{F' \in \mathcal{F}} \|HF'\|_F^2$.

Lemma 3: Let $\mathcal{F} = \{F_1, F_2, \ldots, F_N\}$ be a codebook with packing minimum distance $\delta$ and $N \geq M_t/M_r$. If there exists a vector $v \in \mathbb{C}^{M_t}$ such that $F_k v = 0$ for all $1 \leq k \leq N$, then

1) There exists $(k', l')$ such that $f_{k',l'} = \sum_{j=1}^{M} \alpha_j f_{k_j,l_j}$ where $k_j$ and $l_j$ are indexing sequences with $(k_j, l_j) \neq (k', l')$, $0 < |\alpha_j| < 1$, and $1 \leq m < M_t$.

2) A new codebook $\mathcal{F}$ with

$$
f_{k,i} = \begin{cases} 
\tilde{f}_{k,i} & \text{if } (k, l) \neq (k', l'); \\
v & \text{if } (k, l) = (k', l')
\end{cases}
$$

(12)

has minimum distance $\delta \geq \delta'$.

Proof: The fact that a basis for the column space of the matrix $\tilde{E} = [F_1 F_2 \ldots F_N]$ can be formed from $m < M_t$ columns of the matrix $\tilde{E}$ yields Part 1. The result in Part 2 follows from the fact that for $k \neq k'$,

$$
\|F_k F_{k'}\|_F^2 = \|F_k F_{k'}\|_F \geq \|F_k F_{k'}\|_F \text{ where } F_{k'} = \tilde{f}_{k',1} \ldots \tilde{f}_{k',l'-1} v \tilde{f}_{k',l'+1} \ldots \tilde{f}_{k',M}.
$$

Any $N \geq M_t/M_r$ matrix codebook $\tilde{F}$ with minimum distance $\delta$ and columns $\{f_{k,i}\}$ that does not span $\mathbb{C}^{M_t}$ can be trivially modified to a codebook $F$ with columns $\{f_{k,i}\}$ that span $\mathbb{C}^{M_t}$ and minimum distance $\delta \geq \delta'$ by applying Lemma 3 repeatedly. Therefore, we can state the following theorem.

Theorem 1: If $N \geq M_t/M_r$ then $\mathcal{F} = \{F_1, F_2, \ldots, F_N\}$ provides full diversity order.

Proof: Notice that

$$
\max_{F' \in \mathcal{F}} \|HF'\|_F^2 \geq \frac{1}{M_r} \|HE\|_F^2
$$

(13)

where $E = [F_1 F_2 \ldots F_N]$. If $\{f_{k,i}\}$ spans $\mathbb{C}^{M_t}$, the SVD of $E$ gives $U = [U_L, U_R]$, $U = U(NM, NM)$, and $E$ with diagonal entries $\phi_1 \geq \phi_2 \geq \ldots \geq \phi_{M_t} > 0$ such that $\phi = U_L E U_E^R$. Then using the invariance of complex normal matrices to unitary transformation [19] and the bounds in [18],

$$
\max_{F' \in \mathcal{F}} \|HF'\|_F^2 \geq \frac{1}{M_r} \|E\|_F^2
$$

(14)

The lower bound is the channel gain of an $M_t$ antenna OSTBC system with an array gain shift $\frac{\phi_{M_r}}{NM_r M_t}$. Systems with this effective channel gain obtain a diversity order of $M_t/M_r$ [3]. We can conclude now that the chordal distance limited feedback precoders obtain full diversity order.

Because limited feedback precoding gives full diversity performance, OSTBCs can be generalized to transmit over any transmit antenna configuration with full diversity. This lemma generalizes the result in [7] that proves antenna subset selection OSTBCs achieve full diversity order.

VI. SIMULATIONS

Precoded OSTBCs using chordal distance limited feedback precoders have been simulated in this section. The precoder codebooks were designed using the criterion proposed in Section IV and implemented with codebooks designed from the single antenna non-coherent codes in [20]. For comparison, OSTBCs with antenna subset selection were also simulated [7].

Experiment 1: The first experiment compares antenna subset selection and eight bit chordal distance precoding for a precoded Alamouti code transmitted on an 8 $\times$ 1 wireless system using 16-quadrature amplitude modulation (QAM). The SER performance for an Alamouti code transmitted on a 2 $\times$ 1 system is shown for comparison. The results are given in Fig. 1. Note that antenna subset selection gives an 8dB gain over the two antenna unprecoded system at an error rate of $10^{-3}$. Using eight bit chordal distance precoding provides approximately a 1.4dB gain over antenna subset selection.

Experiment 2: Fig. 2 considers a 4 $\times$ 2 system using a linearly precoded Alamouti code constructed from a 4-QAM constellation. Antenna subset selection provides a 3dB array gain compared with a two antenna unprecoded OSTBC. Limited feedback precoding with a three bit codebook provides over a 0.3dB array gain and with a five bit codebook provides over a 0.7dB array gain over antenna subset selection. Interestingly, antenna subset selection requires $\lceil \log_2 (\frac{4 \times 1}{2}) \rceil = 3$.
error analysis is needed on recent space-time codes and higher rate space-time codes. Thorough probability of obtaining full diversity order.

Using the chordal distance, and proved that the codebooks relate to packing subspaces in the Grassmann manifold using an SER selection criterion and conveys the matrix to the receiver. The receiver chooses the optimal precoding matrix from the codebook a priori to both the transmitter and receiver. The receiver having columns of $I_M$ gives a 0.3dB gain.

bits of feedback. Thus, simply lifting the restriction of $F$ having columns of $I_M$, gives a 0.3dB gain.

![Fig. 1. SER comparison of various OSTBC precoding schemes for a two substream $8 \times 1$ system using 16-QAM.](image1)

![Fig. 2. SER comparison of various OSTBC precoding schemes for a two substream $4 \times 2$ system using 4-QAM.](image2)

VII. CONCLUSIONS AND FUTURE WORK

We proposed a limited feedback precoding technique for OSTBCs that uses a codebook of precoding matrices known a priori to both the transmitter and receiver. The receiver chooses the optimal precoding matrix from the codebook using an SER selection criterion and conveys the matrix to the transmitter over an error-free, zero-delay feedback channel. We addressed the problem of codebook design, showed that it relates to packing subspaces in the Grassmann manifold using the chordal distance, and proved that the codebooks obtain full diversity order.

It is of interest to extend these results to more general and higher rate space-time codes. Thorough probability of error analysis is needed on recent space-time codes codes in order to find simple precoder selection criteria such as the criterion discussed herein. The codebook design problem needs to be readdressed for different space-time codes to find out if Grassmannian subspace packing codebooks still provide performance benefits over other limited feedback methods.

REFERENCES