Grassmannian Beamforming on Correlated MIMO Channels

Abstract—The diversity gains available from multiple-input multiple-output (MIMO) wireless systems are well documented. These gains are realizable through the use of transmit beamforming and receive combining. Transmit beamforming relies on the assumption of channel knowledge at the transmitter, an assumption that is often unrealistic. Limited feedback beamformers have been designed over the past few years, but they concentrate primarily on the assumption of a spatially uncorrelated Rayleigh fading channel matrix. This paper addresses the design of limited feedback beamformers for transmit and receive correlated MIMO channels. In particular, we use a technique where the receiver chooses the beamforming vector from a codebook of possible vectors and conveys this vector over a limited feedback channel. We show how this method obtains full diversity order. Monte Carlo simulations show performance close to optimal beamforming.

I. INTRODUCTION

Multiple-input multiple-output (MIMO) wireless systems have received much interest over the last few years because of their large capacity and diversity gains over single antenna systems [1]. Despite the theoretical capacity benefits, there is still much interest in the diversity-only usage of MIMO systems. There are two main approaches to diversity-only MIMO: space-time coding and beamforming. Of these two schemes, beamforming is the clear winner in terms of probability of error performance because it actually adapts to the channel using closed-loop techniques [2].

Closed-loop methods work by using forwardlink channel knowledge at the transmitter. While this knowledge can sometimes be obtained using forward and reverse channel reciprocity, many systems such as those using frequency division duplexing (FDD) will not have a priori channel knowledge. The usage of limited feedback has been proposed to overcome this lack of channel knowledge in beamforming systems [3]–[7]. These algorithms work by using a codebook of beamforming vectors that is designed off-line, fixed for all channel realizations, and known at both the transmitter and receiver. Under the assumption of spatially uncorrelated Rayleigh fading matrix channels, [4], [5] derive a beamforming design criterion based on thinking of the codebook vectors as points in the Grassmann manifold. Unfortunately, this analysis does not extend to correlated Rayleigh channel models.

This paper analyzes the design of limited feedback beamforming in correlated Rayleigh models. We consider the European Union Information Society Technologies (IST) Multi-Element Transmit and Receive Antennas (METRA) model [8], [9]. This model is a realistic channel model that has been proposed for usage in the IEEE 802.11N study group [10] and studied for usage in the IEEE 802.20 working group on mobile broadband wireless access [11]. This model has also been experimentally validated in [9], [12]. We find that a beamforming codebook design criterion can be obtained by rotating the Grassmannian line packing beamformers studied in [4], [5] by the square root of the transmit correlation matrix. In proving this result, we show that receive correlation does not affect the distribution of the optimal beamforming vector. This means that in the presence of only receive correlation the design criterion in [4], [5] are still applicable. We also show that the optimal beamforming direction is independent of the beamformer channel gain even in the presence of receive correlation. This motivates the feedback of only direction information.

This paper is organized as follows. Section II reviews the beamforming framework for MIMO systems. The effect of receive correlation is analyzed in Section III. A codebook design criterion is developed in Section IV. Section V shows that the proposed criterion can yield full diversity performance. Monte Carlo beamforming simulations results are presented in Section VI. We conclude in Section VII.

II. SYSTEM OVERVIEW

Transmitters employing sufficiently spaced multiple antennas allow signals to be designed over the dimension of space. A beamforming $M_t$ antenna transmitter works by transmitting an $M_t$-dimensional complex vector

$$x_k = ws_k$$
Thus we will restrict \( \parallel \) using beamforming and optimal coherent maximum likelihood given by sponds to maximizing the receive signal-to-noise ratio (SNR) w knowledge, will choose \( F \) the receiver to the transmitter using \( \parallel \)ing techniques design therein to a wide range of more realistic system models. Unfortunately, we have still not answered the question of dealing with transmit correlation.

The design of the codebook \( F \) will be dependent on the distribution of the optimal, unquantized beamforming vector \( w_{\text{opt}} \). This vector is actually the right singular vector of \( H \) corresponding to the largest singular value and sets \( \parallel H w_{\text{opt}} \parallel^2_2 = \parallel H \parallel^2_2 \). Note that this vector is not unique because \( \parallel H e^{j\theta} w_{\text{opt}} \parallel^2_2 = \parallel H w_{\text{opt}} \parallel^2_2 \) for any \( \theta \in [0, 2\pi] \).

### III. Effect of Receive Correlation

We will first analyze the effect of receive only correlation. This analysis will be instrumental in understanding the fully correlated case. This analysis also serves as an extension to the work in [4], [5].

Consider the matrix given by \( \tilde{H} = R_R G \) with \( R_R \) and \( G \) defined as in (2). Let the singular value decomposition be denoted as \( \tilde{H} = U_L A U_R^* \) where \( U_L \) is an \( M_t \times M_t \) unitary matrix, \( U_R \) is an \( M_t \times M_t \) unitary matrix, \( \Lambda \) is a real diagonal matrix with decreasing diagonal entries \( \lambda_1 \geq \cdots \geq \lambda_{\min(M_t, M_r)} \), and \( * \) denotes conjugation and transportation.

The beamforming structure of the MIMO system is completely specified with knowledge of \( \Lambda \) and \( U_R \) at the transmitter. The optimal beamforming vector is given by \( w_{\text{opt}} = U_R [1 \ 0 \ \cdots \ 0]^T \) with channel gain \( \lambda_1^2 \). The following Lemma characterizes both the distribution of \( U_R \) and its relationship to \( \Lambda \).

**Lemma 1:** \( U_R \) is a uniformly distributed unitary matrix that is independent of \( \Lambda \) if \( G \) has i.i.d. \( \mathcal{CN}(0, 1) \) entries.

**Proof:** If \( G \) is a spatially uncorrelated complex normal matrix, its distribution does not change if pre- or post-multiplied by a fixed unitary matrix [14]. Thus the matrix \( \tilde{H} \) is distribution invariant to post-multiplication by any fixed unitary matrix. Thus \( \tilde{H} \) is a right-rotationally invariant random matrix. The singular value matrices \( \Lambda \) and \( U_R \) are actually defined by the eigenvalue decomposition \( \tilde{H}^* \tilde{H} = U_R \Lambda^2 U_R^* \).

The matrix in (5) is isotropically random [15]. Thus the matrix \( U_R \) is uniformly distributed on the group of unitary matrices and is independent of \( \Lambda \).

The ramifications of Lemma 1 are that the codebooks and analysis in [4], [5] are still valid even when the receive antenna array is correlated. This greatly expands both the contribution and applicability of the Grassmannian beamforming techniques design therein to a wide range of more realistic system models. Unfortunately, we have still not answered the question of dealing with transmit correlation.

The vector \( \mathbf{w} \) will be restricted to lie in a codebook \( \mathcal{F} = \{ f_1, f_2, \ldots, f_N \} \). The receiver, which has full channel knowledge, will choose \( \mathbf{w} \) as a function of the current channel realization in order to minimize the instantaneous probability of error and maximize the instantaneous capacity. This corresponds to maximizing the receive signal-to-noise ratio (SNR) given by

\[
\gamma_r = \frac{\| \mathbf{H} \mathbf{w} \|^2_2 \mathcal{E}}{N_0} = \Gamma_r \mathcal{E}.
\]

Because \( \mathcal{E} \) and \( N_0 \) are fixed, this corresponds to choosing

\[
\mathbf{w} = \arg\max_{\mathbf{f} \in \mathcal{F}} \| \mathbf{H} \mathbf{f} \|^2_2. \tag{4}
\]

The optimization in (4) can be implemented by simply evaluating the cost function over each of the \( N \) codebook vectors. Once the vector is chosen, it can be conveyed from the receiver to the transmitter using \( \lfloor \log_2 N \rfloor \) bits of feedback.

As in [13], we will call the term \( \Gamma_r \) in (3) the effective channel gain of the effective single-input single-output channel created through beamforming and receive detection. Note that the average transmitted symbol power is given by \( \| \mathbf{w} \|^2_2 \mathcal{E} \). Thus we will restrict \( \| \mathbf{w} \|^2_2 = 1 \) in order to enforce a transmit power constraint. This means that the codebook \( \mathcal{F} \) is a finite set of unit vectors.
IV. CODEBOOK DESIGN CRITERION

We will now consider the fully correlated channel model in (2). We will make use of the results in Section III during this analysis.

Assuming the full correlation model of (2), the optimal beamforming vector will yield $||Hw_{opt}||^2_2 = ||H||^2_2$. We will take the “communications vector quantization” approach of [4] to define a system parameter and try to minimize the loss in that system parameter due to quantization. Because improving the probability of error and capacity both relate to maximizing $\Gamma$, we will attempt to minimize the loss in our channel gain due to quantization. Thus, we consider a distortion function

$$D(F) = E_H [||H||^2_2 - ||Hw||^2_2]$$

with $w$ chosen according to (4).

We can bound the distortion in (6) as

$$E_H [||H||^2_2 - ||Hw||^2_2] = E_H [||H||^2_2 - ||HR_Tw||^2_2]$$

$$\leq E_H [||H||^2_2 - \lambda^2_I \sum_{j=I}^{N} \frac{1}{\lambda^2_j} ||R_{T}w||^2_2]$$

$$= E_H [||H||^2_2] - E_H [\lambda^2_I] \cdot E_H [\lambda^2_I R_T w^2_2]$$

(7)

$$E_H [\lambda^2_I R_T w^2_2]$$

where (7) is a result of zeroing out all but the largest singular values of $H$, $w_{opt}$ is the dominant right singular vector of $H$, and (8) follows from Lemma 1. This distortion only has one term that depends on the form of the codebook $F$. Thus, we would like to quantize the vector $R_T w_{opt}$.

The modified correlation $\sum_{j=I}^{N} \frac{1}{\lambda^2_j} ||R_T w||^2_2$ is a subspace correlation because it only depends on the column space of $w_{opt}$ and $w$. The column space of a vector is a line. Thus just as in [4], [5] we should think of $F$ as a set of lines rather than a set of vectors. The set of lines in the $M_t$-dimensional vector space is called the Grassmannian manifold $G(M_t, 1)$ [4]. A distance between points on $G(M_t, 1)$ can be defined by the sine of the angle between the corresponding lines in $C^{M_t}$ [16]. For example, if $P_1, P_2 \in G(M_t, 1)$ with corresponding unit vectors $v_1, v_2 \in C^{M_t}$ then

$$d(P_1, P_2) = \sqrt{1 - |v_1^* v_2|^2}$$

(9)

We will use a technique called companding to compress $R_T w_{opt}$. The basic idea of companding is to transform a source for quantization and then transform it back before usage [17]. We will design our codebook using ideas from the companding literature.

Our compander will work by designing a quantizer for $w_{opt}$ obtained by inverting the square root of the correlation matrix. Codebooks for uncorrelated MIMO beamformers were designed in [4] and found to relate to the problem of Grassmannian line packing. Grassmannian line packings would design a codebook $C = \{c_1, c_2, \ldots, c_N\}$ such that

$$\min_{k \neq l} d(P_{c_k}, P_{c_l})$$

where $d$ is defined as in (9) and $P_{c_k}$ is the column space of $c_k$.

A companding approach would then use the codebook obtained from multiplying each element of $C$ by $R_T^*$. This approach, however, does not maintain our unit vector power constraint requirement. Thus we will normalize our codebook to enforce the unit vector requirement.

Therefore to perform the codebook design, we will take a two-step approach of designing a codebook for a spatially uncorrelated source and then rotating our codebook. This leads to the following criterion.

**Correlated Grassmannian Beamforming:** Design $F$ by picking $\{e_1, e_2, \ldots, e_N\}$ that maximize

$$\delta = \min_{1 \leq k < l \leq N} \sqrt{1 - |c_k^* c_l|^2}$$

and setting

$$f_i = \frac{R_T^* c_i}{||R_T^* c_i||_2}$$

This criterion actually allows the system to adjust the limited feedback beamforming to current correlation conditions. The transmitter spatial correlation matrix can usually be estimated at the transmitter without any additional feedback. This would allow only one beamforming codebook, designed using the Grassmannian beamforming criterion in [4], to be stored in a system because the same codebook could be adapted to any transmit and receive correlation structure.

V. DIVERSITY PERFORMANCE

The error rate reduction benefits of MIMO systems are usually characterized in terms of *diversity gain*. The diversity gain of a MIMO system is the asymptotic log-log slope of the probability of error curve. More precisely, a system is said to have diversity gain $d$ if [1]

$$d = \lim_{SNR \to \infty} \log \frac{P_e(SNR)}{\log \text{SNR}}$$

Specifically, we should be able to determine if the designed beamforming vectors provide full diversity order for any form of transmit and receive correlation. The following theory answers this important question.

**Theorem 1:** A Grassmannian beamforming system employing beamforming and combining over correlated Rayleigh fading channels provides full diversity order if the vectors in the beamforming codebook span $C^{M_t}$.

**Proof:** Suppose that the beamforming vectors span $C^{M_t}$. Because there are only $M_t, M_r$ independently fading parameters, the diversity order is bounded by $M_t M_r$. We can construct an invertible matrix $B = R_T F = R_T [f_{k_1}, f_{k_2}, \ldots, f_{k_{M_t}}]$ where $1 \leq k_i \leq N$ for all $i$. Since the matrix is invertible, we can define a singular value decomposition

$$B = V_L \Phi V_R^*$$

(10)
where $V_L$ and $V_R$ are $M_t \times M_t$ unitary matrices and $\Phi$ is a diagonal matrix with diagonal entries $\phi_1 \geq \phi_2 \geq \ldots \geq \phi_{M_t} > 0$.

We have thus proven that the system achieves a diversity order $d$ and (14) follows from the matrix norm bounds described in notes equivalence in distribution. Eq. (11) used the invariance transmit correlation. The results are shown in Fig. 1. The against antenna selection [19], codebooks designed in [4], and spatially correlated Rayleigh fading SD system with array gain shift of $\phi_\text{sd}$. This is the probability of symbol error for an uncorrelated Rayleigh fading SD system with array gain shift of $\phi_\text{sd}$ which provides a diversity of order $M_t M_r$. We have thus proven that the system achieves a diversity order of $M_t M_r$.

VI. SIMULATIONS

Correlated Grassmannian beamforming was simulated to against antenna selection [19], codebooks designed in [4], and unquantized maximum ratio transmission [20].

Experiment 1: This experiment addresses the probability of symbol error for a $4 \times 5$ beamforming system with only transmit correlation. The results are shown in Fig. 1. The correlation matrices were taken from the “Micro Correlated” measurements in [12]. The limited feedback codebooks were restricted to six bits of feedback. Unrotated beamforming outperforms antenna selection by 4.5dB. Note that simply rotating and normalizing the codebooks from [4] yields performance approximately the same as full channel knowledge beamforming. Both unquantized and rotated Grassmannian beamforming perform around 0.6dB better than unrotated beamforming.

Experiment 2: The second experiment was for a transmit and receive correlation $3 \times 2$ MIMO system using four bits of feedback. The correlation matrices were generating using the IEEE 802.11N techniques outlined in [21] with a $5^\circ$ angle spread and three clusters. The results are shown in Fig. 2. Interestingly, the rotated and normalized approach provides performance halfway between the unrotated codebook design and the full channel knowledge case.

Experiment 3: The final experiment focused on a receive-only correlation setting. We considered a $3 \times 4$ system with the receive correlation matrix taken from the “Pico Decorrelated” measurements in [12]. Five bits of feedback were considered. The simulation results are shown in Fig. 3. Grassmannian beamforming performs within 0.6dB of the full channel knowledge case.

VII. SUMMARY AND CONCLUSIONS

We proposed a limited feedback beamforming technique for spatially correlated Rayleigh fading channels. This new method is actually a simple modification to the existing limited feedback beamforming (or Grassmannian beamforming) method discussed in [4]. The new codebooks are only required to be rotated and normalized versions of the traditional codebooks. Monte Carlo simulation results show that this
new approach provides large performance gains over limited feedback beamformers designed for uncorrelated channels.

Note that in this paper we have only considered a rotation based approach to the correlated design. It would be of interest to derive the optimal design strategy to maximize performance. It is likely that an optimal design would have a difficult codebook design that would make the technique impractical. It is also of interest to find efficient ways to search over these subspace codebooks. Currently, only a brute force search is used. It might be possible to use other coding techniques to localize the beamforming vector required for feedback to a small search sphere in the Grassmann manifold.

Fig. 2. Probability of symbol error performance comparison on a $3 \times 2$ transmit and receive correlated channel MIMO system.

Fig. 3. Probability of symbol error performance comparison on a $3 \times 4$ MIMO system with receive correlation.

REFERENCES