$H(Y^N|U_c) = H(Y^N|U^N)$ and $H(Z^N|U_c) = H(Z^N|U^N)$; (5) follows from the i.i.d. properties of U^N, Y^N and Z^N.

In Step 2), we note that $R - \epsilon = I(\tilde{U};\tilde{Y}) - I(\tilde{U};\tilde{Z}) - \epsilon$ and therefore $(R-\epsilon)(1-\epsilon/2)$ is parabolic in ϵ, with the minimum at $\epsilon = 1 + [I(\tilde{U};\tilde{Y}) - I(\tilde{U};\tilde{Z})]/2$. Furthermore, $(R-\epsilon)(1-\epsilon/2) = I(\tilde{U};\tilde{Y}) - I(\tilde{U};\tilde{Z})$ when $\epsilon = 0$ or $2 + I(\tilde{U};\tilde{Y}) - I(\tilde{U};\tilde{Z})$. Hence, $(R-\epsilon)(1-\epsilon/2) < I(\tilde{U};\tilde{Y}) - I(\tilde{U};\tilde{Z})$ when $\epsilon \in (0, 2 + I(\tilde{U};\tilde{Y}) - I(\tilde{U};\tilde{Z}))$.

On the other hand, for $\delta > 0$

$$I(U;Y) - I(U;Z) < I(\tilde{U};\tilde{Y}) - I(\tilde{U};\tilde{Z}). \quad (16)$$

Since $I(U;Y) - I(U;Z)$ is continuous in δ with the value of $I(\tilde{U};\tilde{Y}) - I(\tilde{U};\tilde{Z})$ at $\delta = 0$, and $(R-\epsilon)(1-\epsilon/2)$ is continuous and constant in δ with the value less than $I(\tilde{U};\tilde{Y}) - I(\tilde{U};\tilde{Z})$. If the two curves do not intersect, then there is no constraint on δ. However, if the two curves intersect, then $0 < \delta < \delta^*$, where δ^* is the smallest value such that the two curves intersect. Then the condition on δ becomes $0 < \delta < \delta_0$, where $\delta_0 = \min\{\delta^*, \infty\}$. Thus

$$I(U;Y) - I(U;Z) \geq (R-\epsilon)(1-\epsilon/2)$$

when

$$\epsilon \in (0, 2 + I(\tilde{U};\tilde{Y}) - I(\tilde{U};\tilde{Z}))$$
$$\delta < \delta_0.$$

Note that for $\epsilon \geq 2 + I(\tilde{U};\tilde{Y}) - I(\tilde{U};\tilde{Z}), I(\tilde{U};\tilde{Y}) - I(\tilde{U};\tilde{Z}) - \epsilon < 0$, $1 - \epsilon < 0$, $\epsilon > 1$, and there is nothing to prove.

In step 3, the wiretapper's decoding process is considered. The term $H(U_c|S^K, Z^N)$ is the entropy of the auxiliary codeword given the wiretapper's observation and the bin in which the auxiliary codeword is. It can be bounded by using Fano's inequality as follows:

$$H(U_c|S^K, Z^N) \leq h(P_B) + P_B N I(\tilde{U};\tilde{Z}),$$

where $h(\cdot)$ is the binary entropy function, and P_B is the probability of error in decoding Z^N for U_c given S^K, and therefore

$$\frac{H(U_c|S^K, Z^N)}{H(S^K)} \leq \frac{h(P_B) + P_B N I(\tilde{U};\tilde{Z})}{(R-\epsilon)N}. \quad (17)$$

The conditional entropy can be bounded by via bounding the wiretapper's probability of error in the bin decoding. P_B can be made arbitrarily small given sufficiently small δ and sufficiently large N via the AEPs since there are $2^{N[I(\tilde{U};\tilde{Z}) - \epsilon_{UZ}]}$ sequences in a subbin which is exponentially smaller than $2^{NI(\tilde{U};\tilde{Z})}$.

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Feedback Rate-Capacity Loss Tradeoff for Limited Feedback MIMO Systems

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***Abstract*—Multiple-input–multiple-output (MIMO) communication systems can provide large capacity gains over traditional single-input–single-output (SISO) systems and are expected to be a core technology of next generation wireless systems. Often, these capacity gains are achievable only with some form of adaptive transmission. In this paper, we study the capacity loss (defined as the rate loss in bits/s/Hz) of the MIMO wireless system when the covariance matrix of the transmitted signal vector is designed using a low rate feedback channel. For the MIMO channel, we find a bound on the ergodic capacity loss when random codebooks, generated from the uniform distribution on the complex unit sphere, are used to convey the second order statistics of the transmitted signal from the receiver to the transmitter. In this case, we find a closed-form expression for the ergodic capacity loss as a function of the number of bits fed back at each channel realization. These results show that the capacity loss decreases at least as $O\left(2^{-B/(2MM_t-2)}\right)$ where B is the number of feedback bits, M_t is the number of transmit antennas, and $M = \min\{M_r, M_t\}$ where M_r is the number of receive antennas. In the high SNR regime, we present a new bound on the capacity loss that is tighter than the previously derived bound and show that the capacity loss decreases exponentially as a function of the number of feedback bits.**

***Index Terms*—Adaptive modulation, capacity loss, limited feedback, multiple-input–multiple-output (MIMO) systems, Rayleigh channels.**

I. Introduction

Because of their capacity and quality benefits, multiple-input–multiple-output (MIMO) wireless systems are expected to be a core technology in next evolution third-generation (3G) and fourth-generation (4G) wireless systems. In addition, the performance of MIMO systems can be significantly improved by adapting the transmitted signal to the current channel conditions (see, for example, the discussion in [1]). When the channel cannot be estimated at the transmitter, such as is the case in frequency division duplexing, systems can employ a feedback link to convey quantized channel state information (CSI) and obtain capacity performance close to the scenario when the transmitter perfectly knows the channel. The feedback rate, however, must be chosen judiciously because the feedback channel may only support a small data rate and the feedback bits are allocated as overhead on the reverse data path. To satisfy these rate constraints, low-rate (or limited) feedback has been studied in various scenarios and special cases [2]–[18]. These techniques include feedback adaptation techniques specific to transmit beamforming [6]–[8], precoded orthogonal space-time block coding [9], [11], [12], [16], and precoded spatial multiplexing [14]. Initial performance analysis of some of these techniques was given in [8], [9], and [13]. The basic idea is that a limited number of feedback bits representing some sort of CSI are transmitted from the receiver to the transmitter. The transmitter uses this small number of bits to adapt

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the transmitted signal to the current channel conditions. These limited feedback systems are currently being studied for use in IEEE 802.11N and IEEE 802.16E compliant wireless systems.

One particularly powerful adaptive technique is covariance adaptation [3], [4], [19], [20]. In this technique, feedback is used to convey the covariance matrix to the transmitter. Covariance adaptation has been considered for unquantized statistical feedback [19], [21] and for limited feedback [3], [4]. Other work has considered random coding techniques [13], antenna selection techniques [22], [23], and per antenna rate control [24], [25]. When covariance adaptation is combined with limited feedback, the transmitted signal's covariance matrix is restricted to lie in a codebook that is known to both the transmitter and receiver. The receiver can then choose the optimal covariance matrix from the codebook using its channel estimate and transmit the binary index of the chosen covariance matrix to the transmitter. Because the transmitter knows the codebook, the received binary index can be used to recover the coodebook covariance matrix chosen based on current channel conditions.

Despite the expected multiplexing and diversity gains when feedback is employed in MIMO systems, feedback still requires increased transmitter and receiver complexity. Therefore, it is of utmost importance to quantify the inherent tradeoff between data rate and feedback amount. Characterizing this tradeoff is equivalent to understanding the relationship between the *capacity loss* incurred from using finite instead of infinite resolution feedback and the *feedback rate*, (i.e., the bits/s/Hz reduction incurred from using a limited number of feedback bits instead of perfect feedback). To remedy this problem, we analyze the rate degradation or capacity loss incurred by restricting the transmitted signal's covariance matrix to lie in a codebook known to both the transmitter and receiver. We analytically tradeoff the feedback rate and capacity loss in the same manner as quantization rate and distortion are tradedoff in rate-distortion theory [26]. Our results show that the capacity loss decreases as $O\left(2^{-B/(2MM_t-2)}\right)$ as the number of feedback bits B grows large for M_t transmit antennas, M_r receive antennas, and $M = \min\{M_r, M_t\}$.

We assume that the matrices in the covariance codebook are generated independently and uniformly on the set of positive semi-definite, Hermitian, trace constrained matrices. Interestingly, we show this uniform distribution can be generated from the uniform (or Haar) distribution on a higher-dimensional complex unit sphere. The uniform complex spherical distribution has been shown to be optimal for covariance quantization in the MISO Rayleigh fading setting [13] and is extremely robust to situations where the channel has rapidly changing statistics. We compute a closed-form expression for the capacity loss as a function of the number of bits for this random codebook method. Note that asymptotic capacity loss results were also derived in [13], [27], [28].

This paper is organized as follows. In Section II, we give an overview of the limited feedback system model under consideration. In Section III, we study uniformly distributed covariance codebooks. In Section IV, we derive a bound on the ergodic capacity loss as a result of using randomly generated codebooks with uniform distribution on the complex unit sphere. Furthermore, in the high SNR regime we derive a tighter upper bound for the ergodic capacity loss, and we discuss the implications of this analysis and the design of feedback as an overhead in a two-way communication system in Section V. In Section VI simulation results are presented, and we conclude with future points of emphasis in Section VII.

II. Limited Feedback System Overview

We consider the flat fading MIMO channel with additive white Gaussian noise. The vector of received samples at block index k is $\boldsymbol{y}_k = \left(y_1^k, \dots, y_{M_r}^k\right)^T$ from the M_r receive antennas and is given by

$$\boldsymbol{y}_k = \boldsymbol{H}_k \boldsymbol{x}_k + \boldsymbol{e}_k$$

where $\boldsymbol{x}_k = \left(x_1^k, \dots, x_{M_t}^k\right)^T$ is the transmitted vector from the M_t transmit antennas, $\boldsymbol{H}_k$ is the $M_r \times M_t$ channel matrix, $\boldsymbol{e}_k = \left(e_1^k, \dots, e_{M_r}^k\right)^T$ is the zero-mean complex Gaussian noise vector with covariance matrix $N_0 \boldsymbol{I}_{M_r}$, where $\boldsymbol{I}_{M_r}$ is the $M_r \times M_r$ identity matrix. The block length will be denoted by K, and the block of transmitted vectors conveys a random message variable $\omega \in \{1, \dots, \lceil 2^{KR} \rceil\}$ where R is the rate.

We will assume a power constraint of

$$\mathrm{tr}\left(E\left[\boldsymbol{x}_k(\omega)\boldsymbol{x}_k^*(\omega)\right]\right) \leq \mathcal{E}_k$$

where $E[\cdot]$ denotes the expectation operation taken over the random message variable ω, $\mathrm{tr}(\cdot)$ denotes the trace of the matrtix, and $\boldsymbol{x}_k^*$ denotes the conjugate transpose of $\boldsymbol{x}_k$. We will use the assumption (as discussed in [29]) that $\mathcal{E}_k = \mathcal{E}$ for all $k \in \{1, 2, \dots, K\}$. Therefore, the power constraint is *independent* of the channel realization and no temporal power control is used.

Two cases of channel conditions will be employed in our discussion and development. The first case will be the case of a fixed and deterministic channel where $\boldsymbol{H}_k = \boldsymbol{H}$ for all $k \in \{1, 2, \dots, K\}$. For the fixed and deterministic case we will assume that $\boldsymbol{H}$ is full rank. The second case will be the independent and identically distributed (i.i.d.) Rayleigh-fading scenario where each $\boldsymbol{H}_k$ is chosen memorylessly to have i.i.d. $\mathcal{CN}(0,1)$ entries.

It is assumed that an instantaneous feedback link is employed in the communication system and that the feedback is noiseless with a limited transmission rate of B bits/cycle. Also, we assume that the receiver has complete knowledge of the channel matrix $\boldsymbol{H}_k$. For each channel realization, B bits of data are conveyed back to the transmitter.

The receiver uses its knowledge of $\boldsymbol{y}_1, \dots, \boldsymbol{y}_K$ and $\boldsymbol{H}_1, \dots, \boldsymbol{H}_K$ to decode to a symbol output realization $\widehat{W}$ of the transmitted message realization W. The probability of error will be denoted as $P_e^{(K)} = Pr(\hat{\omega} \neq \omega)$. If a sequence of $(\lceil 2^{KR} \rceil, K)$ codes exists such that the power constraint is satisfied and $P_e^{(K)} \to 0$ as $K \to \infty$, the rate R is said to be achievable. The supremum of all achievable rates defines the capacity of the system.

For a fixed and deterministic channel realization $\boldsymbol{H}_k = \boldsymbol{H}$ and signal-to-noise ratio $\rho = \mathcal{E}/N_0$, given that the input signal has a complex Gaussian vector distribution, the mutual information of the MIMO channel subject to the covariance constraint $E\left[\boldsymbol{x}_k(\omega)\boldsymbol{x}_k^*(\omega)\right] = \mathcal{E}\boldsymbol{Q}$ is given by [1], [30]

$$C_{\boldsymbol{Q}}(\rho; \boldsymbol{H}) \equiv \log\det\left(\boldsymbol{I}_{M_r} + \rho \boldsymbol{H}\boldsymbol{Q}\boldsymbol{H}^*\right) \tag{1}$$

where $\boldsymbol{Q}$ is the $M_t \times M_t$ covariance matrix used at the transmitter, $\log(\cdot)$ is the base-2 logarithm, and $\det(\boldsymbol{A})$ denotes the determinant of $\boldsymbol{A}$. In the i.i.d. Rayleigh-fading scenario, the ergodic mutual information subject to the same second-order constraint used to derive (1) is defined by

$$C_{\boldsymbol{Q}}(\rho) \equiv E\left[\log\det\left(\boldsymbol{I}_{M_r} + \rho \boldsymbol{H}_k\boldsymbol{Q}\boldsymbol{H}_k^*\right)\right] \tag{2}$$

where the expectation is taken over the distribution of $\boldsymbol{H}_k$.

According to the assumed power constraint, the covariance $\mathcal{E}\boldsymbol{Q}^{(k)} = E\left[\boldsymbol{x}_k(\omega)\boldsymbol{x}_k^*(\omega)\right]$ is only constrained with respect to the transmit power with $\mathrm{tr}(\boldsymbol{Q}^{(k)}) \leq 1$. In this case, the mutual information given in (1) is maximized if the vector $\boldsymbol{x}_k$ has a complex Gaussian distribution, with covariance matrix $\mathcal{E}\boldsymbol{Q}^{(k)}$ that is chosen by waterfilling over the dominant eigenvectors of the MIMO channel $\boldsymbol{H}_k$ with average power constraint $\mathrm{tr}(\boldsymbol{Q}^{(k)}) \leq 1$. This is also the case of full CSIT where the channel state information is fully known to both the transmitter and receiver (i.e., $B = \infty$). This gives a waterfilling (or "informed transmitter") capacity assuming a fixed and deterministic channel of [30], [31]

$$C_{IT}(\rho; \boldsymbol{H}) = \max_{\boldsymbol{Q}:\mathrm{tr}(\boldsymbol{Q})\leq 1} \log\det(\boldsymbol{I}_{M_r} + \rho \boldsymbol{H}\boldsymbol{Q}\boldsymbol{H}^*) \tag{3}$$

where IT denotes that the capacity assumes that the transmitter is "informed" of the channel realization $\boldsymbol{H}$. In this full channel knowledge scenario, the ergodic capacity for i.i.d. Rayleigh fading is given by [30], [31]

$$C_{IT}(\rho) = E\left[\max_{\boldsymbol{Q}^{(k)}:\mathrm{tr}(\boldsymbol{Q}^{(k)})\leq 1} \log\det\left(\boldsymbol{I}_{M_r} + \rho \boldsymbol{H}_k \boldsymbol{Q}^{(k)} \boldsymbol{H}_k^*\right)\right]. \tag{4}$$

When the transmitter only has channel distribution information, the ergodic capacity for the i.i.d. Rayleigh-fading scenario is given by [31]

$$C_{\mathrm{DIST}}(\rho) = \max_{\boldsymbol{Q}:\mathrm{tr}(\boldsymbol{Q})\leq 1} E\left[\log\det\left(\boldsymbol{I}_{M_r} + \rho \boldsymbol{H}_k \boldsymbol{Q} \boldsymbol{H}_k^*\right)\right]. \tag{5}$$

If the channel has a Gaussian distribution with i.i.d. entries and $B = 0$, the maximizing $\boldsymbol{Q}$ in (5) is [30]

$$\boldsymbol{Q} = \frac{1}{M_t}\boldsymbol{I}_{M_t},$$

i.e., independent streams with equal average power transmitted over the different antennas. For a fixed and deterministic channel realization $\boldsymbol{H}$, and when the transmitted signal has a complex vector Gaussian distribution, the mutual information subject to the covariance constraint $\boldsymbol{Q} = \frac{1}{M_t}\boldsymbol{I}_{M_t}$, which is often called the uninformed transmitter case [1], is denoted by C_{UT} and is given by

$$C_{\mathrm{UT}}(\rho;\boldsymbol{H}) \equiv \log\det\left(\boldsymbol{I}_{M_r} + \frac{\rho}{M_t}\boldsymbol{H}\boldsymbol{H}^*\right). \tag{6}$$

We are interested, however, in the nonextremum cases when $0 < B < \infty$. We assume that both the transmitter and receiver have a codebook $\mathcal{Q} = \{\boldsymbol{Q}_1, \boldsymbol{Q}_2, \ldots, \boldsymbol{Q}_N\}$ of covariance matrices where $N = 2^B$ and $\mathrm{tr}(\boldsymbol{Q}_i) \leq 1$, $1 \leq i \leq 2^B$. For each channel realization, B bits of data, which correspond to the binary index of the covariance matrix in the codebook $\mathcal{Q}$ that maximizes the expression given in (1), are transmitted through the feedback link. The capacity of the system transmitting over a fixed and deterministic wireless system using a limited rate feedback channel and codebook $\mathcal{Q}$ is given by [3], [29], [32]

$$C_{\mathrm{Feed}}^{\mathcal{Q}}(\rho;\boldsymbol{H}) = \max_{\boldsymbol{Q}\in\mathcal{Q}} \log\det\left(\boldsymbol{I}_{M_r} + \rho\boldsymbol{H}\boldsymbol{Q}\boldsymbol{H}^*\right).$$

When the channel is i.i.d. Rayleigh fading and we let the codebook $\mathcal{Q}$ be generated randomly for each channel realization, the ergodic channel capacity in this case is given by

$$C_{\mathrm{Feed}}(\rho) = E\left[\max_{\widetilde{\boldsymbol{Q}}\in\mathcal{Q}} \log\det\left(\boldsymbol{I}_{M_r} + \rho\boldsymbol{H}_k\widetilde{\boldsymbol{Q}}\boldsymbol{H}_k^*\right)\right]$$

where the expectation is over the distribution of $\boldsymbol{H}_k$ and $\mathcal{Q}$.

In general, the codebook $\mathcal{Q}$ can be designed using vector quantization techniques such as the Lloyd algorithm [3], [4]. Unfortunately, these kinds of techniques are highly dependent on the distribution of the channel and need to be redesigned each time the antenna configuration or SNR changes. We will deal primarily with a codebook $\mathcal{Q}$ that is generated *randomly*. We will assume that the entries in the codebook $\mathcal{Q}$ are chosen independently according to the uniform distribution on the set of all possible waterfilling matrices. In this case it is assumed that a random codebook of covariance matrices is generated at each channel realization and is revealed simultaneously to both the transmitter and receiver. In practice, this type of random coding can be implemented using similar techniques used in code-devision multiple-access (CDMA) communication systems where both the transmitter and receiver use randomly generated signature waveforms for the transmission of each information symbol [33].

Since it is assumed that the channel is i.i.d (or stationary and ergodic) in time, an alternative way to think about the expectation over the randomly generated codebook of covariance matrices, is to assume the existence of an ensemble of feedback quantizers, where the expectation is taken over this ensemble and in this case we have one realization for each transmitted codeword. In this case $\mathcal{Q}$ stays fixed over the transmission of a codeword and it does not change with each channel realization.

Random coding techniques are commonly used in information theory to prove the existence of codes in channel capacity proofs. The idea is that if a randomly chosen code can provide a desired level of performance then there exists at least one "good" code. Our reasoning is the same. If a random feedback codebook can provide a certain rate then one codebook can be found that will surpass this rate performance. In Section III, we will probabilistically analyze this kind of random codebook in order to understand the performance of MIMO systems with limited feedback.

III. Random Codebooks

In this section, we are interested in analyzing a randomly generated codebook with uniform distribution on the complex unit sphere. For a given fixed point on the complex unit sphere, our goal is to derive an expression for a metric defined on the space of all $M_t \times M_t$ covariance matrices. The metric definition that we consider in this paper is the common Euclidean norm. The reason behind this interest will be shown later in Section IV where we derive capacity loss bounds with this kind of random coding technique.

In the following, we will consider MIMO limited feedback systems that use a codebook $\mathcal{Q}$ that is chosen randomly for each channel realization. Before we analyze this codebook distribution, we will first have to characterize the power constraint properties of the covariance matrices under consideration.

The general waterfilling optimization problem solves for the covariance matrix $\boldsymbol{Q}$ with power constraint $\mathrm{tr}(\boldsymbol{Q}) \leq 1$ that maximizes the expression in (1). Since this matrix is always a positive semi-definite Hermitian matrix and we do not assume any temporal power control, the power constraint inequality can be replaced with an equality. The following lemma shows why this is true.

Lemma 1: Suppose that $\boldsymbol{Q}_a$ is a MIMO covariance matrix with $0 < \mathrm{tr}(\boldsymbol{Q}_a) = \eta < 1$ and $\boldsymbol{Q}_b = \eta^{-1}\boldsymbol{Q}_a$. Let $\boldsymbol{H}$ be an arbitrary $\mathrm{M}_r \times M_t$ complex matrix. For any SNR ρ

$$\log\det\left(\boldsymbol{I}_{M_r} + \rho\boldsymbol{H}\boldsymbol{Q}_a\boldsymbol{H}^*\right) \leq \log\det\left(\boldsymbol{I}_{M_r} + \rho\boldsymbol{H}\boldsymbol{Q}_b\boldsymbol{H}^*\right).$$

Proof: Let $\lambda_1, \ldots, \lambda_{M_r}$ denote the eigenvalues of $\boldsymbol{H}\boldsymbol{Q}_a\boldsymbol{H}^*$. We know

$$\log\det\left(\boldsymbol{I}_{M_r} + \rho\boldsymbol{H}\boldsymbol{Q}_a\boldsymbol{H}^*\right) = \sum_{i=1}^{M_r}\log(1+\rho\lambda_i).$$

Since the eigenvalues are nonnegative and since $\log$ is a monotonically increasing function,

$$\begin{aligned}\log\det\left(\boldsymbol{I}_{M_r} + \rho\boldsymbol{H}\boldsymbol{Q}_a\boldsymbol{H}^*\right) &\leq \sum_{i=1}^{M_r}\log\left(1+\frac{\rho}{\eta}\lambda_i\right)\\ &= \log\det\left(\boldsymbol{I}_{M_r} + \rho\boldsymbol{H}\boldsymbol{Q}_b\boldsymbol{H}^*\right).\end{aligned}$$

□

Therefore when generating random codebooks of covariance matrices, we will restrict ourselves to the set of matrices defined in the following. Let T denote the set of all $M_t \times M_t$ positive semi-definite matrices. Our randomly generated codebook will consist of elements from the set

$$\mathcal{T}_M = \{\boldsymbol{Q} \mid \boldsymbol{Q} \in T, \boldsymbol{Q}^* = \boldsymbol{Q},\ \mathrm{rank}(\boldsymbol{Q}) \leq M,\ \mathrm{tr}(\boldsymbol{Q}) = 1\}$$

where $M = \min\{M_t, M_r\}$.

Let $\mathrm{vec}(\cdot)$ be the operator which stacks the columns of an $M \times M_t$ matrix into a column vector, and let $\mathrm{unvec}(\cdot)$ be the reverse operator. Interestingly, in the following it is shown how the set $\mathcal{T}_M$ can be parameterized as a quotient space. A reference for quotient spaces can be found in [34]. Intuitively speaking, the quotient space is the set of classes of points which satisfy a certain equivalence relation. In other words, in each class there is some relation that is true for all points in that class. As an example, let the space of all n-dimensional unit norm vectors be denoted by Ω_n. We define an equivalence relation between any two unit vectors $\boldsymbol{w}_1 \in \Omega_n$ and $\boldsymbol{w}_2 \in \Omega_n$ by $\boldsymbol{w}_1 \equiv \boldsymbol{w}_2$ if for some $\theta \in [0, 2\pi)$ $\boldsymbol{w}_1 = e^{j\theta}\boldsymbol{w}_2$ ($\boldsymbol{w}_1$ and $\boldsymbol{w}_2$ lie on the same line). The quotient space in this case is the set of all one-dimensional subspaces in the n-dimensional vector space.

Theorem 1: Let Ω_{MM_t} denote the MM_t-dimensional complex unit sphere, and let $U(M)$ denote the group of unitary $M \times M$ matrices. Then, the set

$$\mathcal{T}_M = \Omega_{MM_t}/U(M)$$

where the quotient space is defined with respect to the equivalence relation $\boldsymbol{w}_1$ is equivalent to $\boldsymbol{w}_2$ if

$$\mathrm{unvec}(\boldsymbol{w}_1)^*\mathrm{unvec}(\boldsymbol{w}_1) = \mathrm{unvec}(\boldsymbol{w}_2)^*\mathrm{unvec}(\boldsymbol{w}_2)$$

with $\boldsymbol{w}_1, \boldsymbol{w}_2 \in \Omega_{MM_t}$. Moreover, there exists a one-to-one map $f : \Omega_{MM_t} \to \mathcal{T}_M \times U(M)$.

This theorem is proved in Appendix.

Corollary 1: Any covariance matrix $\boldsymbol{Q} \in \mathcal{T}_M$ can be written as $\boldsymbol{Q} = \boldsymbol{F}^*\boldsymbol{F}$ where $\boldsymbol{F} = \mathrm{unvec}(\boldsymbol{w})$ is the $M \times M_t$ matrix obtained from a corresponding complex unit vector $\boldsymbol{w} \in \Omega_{MM_t}$.

This corollary follows directly from Theorem 1. Let $\|A\|_2$ denote the 2-norm of A that is defined as the maximum singular value of an $n \times m$ matrix A. Then from the above corollary, a uniformly[1] distributed random covariance codebook $\mathcal{Q}$ with cardinality $N = 2^B$ can be generated in the following way:

1) Generate a random complex Gaussian vector $\boldsymbol{v}$ with MM_t i.i.d. elements.
2) Let $\boldsymbol{w}$ be the normalized random vector $\boldsymbol{v}$, i.e., $\boldsymbol{w} = \boldsymbol{v}/\|\boldsymbol{v}\|_2$.
3) Generate the $M \times M_t$ matrix $\boldsymbol{F}$ from $\mathrm{unvec}(\boldsymbol{w})$.
4) Set $\boldsymbol{Q}_i = \boldsymbol{F}^*\boldsymbol{F}$.
5) Repeat steps 1) to 4) for each $1 \le i \le N$.

Given an arbitrary covariance matrix $\boldsymbol{Q}$ with $\mathrm{tr}(\boldsymbol{Q}) = 1$, we are interested in finding a bound on the mean value of some distortion measure when a uniformly distributed random codebook is used. The distortion measure in this case is defined by the Euclidean distance (or the Frobenius norm) between any two matrices. Hence, we should bound the following:

$$E\left[\min_{1\le i\le N} \|\boldsymbol{Q} - \boldsymbol{Q}_i\|_F\right] \tag{7}$$

where the minimization is over all codebook elements in $\mathcal{Q}$ that are generated randomly as in steps 1)–5) above.

As we mentioned before, we are interested in MIMO communication systems employing feedback with quantized CSI, where for each channel realization a random codebook is generated using a complex spherical uniform distribution. The optimal codebook covariance matrix index that maximizes the mutual information assuming i.i.d. complex Gaussian signaling is sent over the feedback link. In order to analyze the performance of such codebooks, we must bound the ergodic capacity loss incurred by this type of codebook. First, we will look at the average Euclidean distance of (7).

[1]The distribution is uniform in the sense that it is generated by a Haar or uniform distribution on the unit sphere.

Lemma 2: Let Ω_k denote the k-dimensional complex unit sphere, and let $\boldsymbol{v}$ be an arbitrary k-dimensional unit vector $\boldsymbol{v} \in \Omega_k$. Let us also define the set of random variables

$$\psi_i \triangleq \sqrt{1 - |\boldsymbol{v}^*\boldsymbol{w}_i|^2}, \quad 1 \le i \le N \tag{8}$$

and

$$\Psi \triangleq \min_{1\le i\le N} \psi_i$$

where $\{\boldsymbol{w}_i,\ 1 \le i \le N\}$ is a set of N i.i.d. k-dimensional random vectors with a uniform distribution on the complex unit sphere. The probability density function of Ψ is given by

$$f_\Psi(\delta) = 2N(k-1)(1-\delta^{2(k-1)})^{N-1}\delta^{2(k-1)-1}. \tag{9}$$

Proof: By the above definitions

$$\begin{aligned}\mathrm{Prob}(\Psi < \delta) &= \mathrm{Prob}\left(\bigcup_{i=1}^{N} \psi_i < \delta\right)\\ &= 1 - \mathrm{Prob}\left(\bigcap_{i=1}^{N} \psi_i \ge \delta\right).\end{aligned}$$

Since $\{\psi_i, 1 \le i \le N\}$ is a set of i.i.d. random variables

$$\begin{aligned}\mathrm{Prob}(\Psi < \delta) &= 1 - \prod_{i=1}^{N} \mathrm{Prob}(\psi_i \ge \delta)\\ &= 1 - [\mathrm{Prob}(\psi_1 \ge \delta)]^N.\end{aligned}$$

For an arbitrary unit vector $\boldsymbol{v}$ and $\boldsymbol{w}_1$ that is uniformly distributed on Ω_k, we know that [6]

$$\mathrm{Prob}(\psi_i < \delta) = \delta^{2(k-1)}, \quad 1 \le i \le N. \tag{10}$$

Therefore

$$\mathrm{Prob}(\Psi < \delta) = 1 - \left[1 - \delta^{2(k-1)}\right]^N, \tag{11}$$

and by taking the derivative of the last expression with respect to δ we get (9). □

Note that in the sequel it will be assumed that $k = MM_t$.

Lemma 3: If $\boldsymbol{A}$ and $\boldsymbol{B}$ are any two $M \times M_t$ matrices

$$\begin{aligned}&\|\boldsymbol{A}^*\boldsymbol{A} - \boldsymbol{B}^*\boldsymbol{B}\|_F\\ &\qquad \le \sqrt{M}\,\|\mathrm{vec}(\boldsymbol{A})\mathrm{vec}(\boldsymbol{A})^* - \mathrm{vec}(\boldsymbol{B})\mathrm{vec}(\boldsymbol{B})^*\|_F.\end{aligned} \tag{12}$$

Proof: Let a_{jm} denote entry (j, m) of $\boldsymbol{A}$ and b_{jm} denote entry (j, m) of $\boldsymbol{B}$. Using this notation

$$\begin{aligned}&\|\boldsymbol{A}^*\boldsymbol{A} - \boldsymbol{B}^*\boldsymbol{B}\|_F^2\\ &= \sum_{m=1}^{M_t}\sum_{n=1}^{M_t}\left|\sum_{j=1}^{M}\left(a_{jm}^*a_{jn} - b_{jm}^*b_{jn}\right)\right|^2\\ &\le \sum_{m=1}^{M_t}\sum_{n=1}^{M_t}\left(\sum_{j=1}^{M}\left|\left(a_{jm}^*a_{jn} - b_{jm}^*b_{jn}\right)\right|\right)^2\\ &\le \sum_{m=1}^{M_t}\sum_{n=1}^{M_t} M\left(\sum_{j=1}^{M}\left|\left(a_{jm}^*a_{jn} - b_{jm}^*b_{jn}\right)\right|^2\right)\\ &\le M\sum_{m=1}^{M_t}\sum_{n=1}^{M_t}\sum_{j=1}^{M}\sum_{k=1}^{M}\left|\left(a_{jm}^*a_{kn} - b_{jm}^*b_{kn}\right)\right|^2\\ &= M\,\|\mathrm{vec}(\boldsymbol{A})\mathrm{vec}(\boldsymbol{A})^* - \mathrm{vec}(\boldsymbol{B})\mathrm{vec}(\boldsymbol{B})^*\|_F^2.\end{aligned}$$ □

For the distortion measure given in (7), we have the following theorem.

$$\sqrt{2M}N\frac{\Gamma\left(\frac{2k-1}{2k-2}\right)\Gamma(N)}{\Gamma\left(\frac{2k-1}{2k-2}+N\right)} \approx \sqrt{2M}N\frac{\Gamma\left(\frac{2k-1}{2k-2}\right)\sqrt{2\pi}(N-1)^{\left(N-\frac{1}{2}\right)}\exp(-N+1)}{\sqrt{2\pi}\left(N+\frac{1}{2k-2}\right)^{\left(N+\frac{1}{2k-2}+\frac{1}{2}\right)}\exp\left(-N-\frac{1}{2k-2}\right)}$$
$$< \sqrt{2M}N\frac{\Gamma\left(\frac{2k-1}{2k-2}\right)N^{\left(N-\frac{1}{2}\right)}\exp(-N+1)}{N^{\left(N+\frac{1}{2k-2}+\frac{1}{2}\right)}\exp\left(-N-\frac{1}{2k-2}\right)} \tag{17}$$
$$= \sqrt{2M}\Gamma\left(\frac{2k-1}{2k-2}\right)\exp\left(\frac{2k-1}{2k-2}\right)\frac{N^{\left(N+\frac{1}{2}\right)}}{N^{\left(N+\frac{1}{2}+\frac{1}{2k-2}\right)}}. \tag{18}$$

Theorem 2: If $\boldsymbol{Q}$ is an arbitrary $M_t \times M_t$ positive semi-definite matrix such that $\boldsymbol{Q} \in \mathcal{T}_M$ and $\mathcal{Q} = \{\boldsymbol{Q}_1, \boldsymbol{Q}_2, \ldots, \boldsymbol{Q}_N\}$ is a uniformly distributed random codebook

$$E\left[\min_{1\le i\le N}\|\boldsymbol{Q}-\boldsymbol{Q}_i\|_F\right] \le \sqrt{2M}N\frac{\Gamma\left(\frac{2k-1}{2k-2}\right)\Gamma(N)}{\Gamma\left(\frac{2k-1}{2k-2}+N\right)} \tag{13}$$

where N is the number of elements in the codebook, $k = MM_t$, and Γ is the Gamma function given by

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt.$$

Proof: By the corollary of Theorem 1, let $\boldsymbol{F}$ and $\boldsymbol{F}_i$ be defined such that $\boldsymbol{Q} = \boldsymbol{F}^*\boldsymbol{F}$ and $\boldsymbol{Q}_i = \boldsymbol{F}_i^*\boldsymbol{F}_i$. Thus,

$$\|\boldsymbol{Q}-\boldsymbol{Q}_i\|_F = \|\boldsymbol{F}^*\boldsymbol{F}-\boldsymbol{F}_i^*\boldsymbol{F}_i\|_F.$$

Using Lemma 3

$$\|\boldsymbol{Q}-\boldsymbol{Q}_i\|_F \le \sqrt{M}\,\|\mathrm{vec}(\boldsymbol{F})\mathrm{vec}(\boldsymbol{F})^* - \mathrm{vec}(\boldsymbol{F}_i)\mathrm{vec}(\boldsymbol{F}_i)^*\|_F. \tag{14}$$

Now let $\boldsymbol{v} = \mathrm{vec}(\boldsymbol{F})$ and $\boldsymbol{w}_i = \mathrm{vec}(\boldsymbol{F}_i)$, then the right hand side of (14) is just the chordal distance [35], [36] between the column spaces (i.e., lines) generated by two complex unit vectors $\boldsymbol{v}$ and $\boldsymbol{w}_i$ scaled by a factor $\sqrt{2M}$. Therefore

$$\|\boldsymbol{v}\boldsymbol{v}^* - \boldsymbol{w}_i\boldsymbol{w}_i^*\|_F = \sqrt{2}\sqrt{1-|\boldsymbol{v}^*\boldsymbol{w}_i|^2}. \tag{15}$$

From Lemma 2

$$\begin{aligned} E&\left[\min_{1\le i\le N}\|\boldsymbol{Q}-\boldsymbol{Q}_i\|_F\right] \\ &\le E\left[\min_{1\le i\le N}\sqrt{2M}\sqrt{1-|\boldsymbol{v}^*\boldsymbol{w}_i|^2}\right] \\ &= \sqrt{2M}\int_0^1 x f_\Psi(x)dx \\ &= \sqrt{2M}\int_0^1 x\ 2N(k-1) \\ &\quad \times (1-x^{2(k-1)})^{N-1}x^{2(k-1)-1}dx. \end{aligned} \tag{16}$$

By the result in [37, p. 322, eq. 3.251], (16) can be expressed as

$$E\left[\min_{1\le i\le N}\|\boldsymbol{Q}-\boldsymbol{Q}_i\|_F\right] \le \sqrt{2M}N\frac{\Gamma\left(\frac{2k-1}{2k-2}\right)\Gamma(N)}{\Gamma\left(\frac{2k-1}{2k-2}+N\right)}. \qquad \square$$

Note that the bound in (13) goes to zero as the number of bits used in the codebook goes to infinity. This can be verified easily if we roughly apply Stirling's approximation for the Γ function yielding (17) and (18) shown at the top of the page. If

$$\beta_k \equiv \Gamma\left(\frac{2k-1}{2k-2}\right)\exp\left(\frac{2k-1}{2k-2}\right)$$

then substituting (18) into (13) and 2^B for N we get

$$\begin{aligned} E\left[\min_{1\le i\le 2^B}\|\boldsymbol{Q}-\boldsymbol{Q}_i\|_F\right] &\le \sqrt{2M}\beta_k 2^{-\frac{B}{2k-2}} \\ &\le 4\sqrt{2M}\,2^{-\frac{B}{2k-2}} \end{aligned} \tag{19}$$

where the last inequality holds because β_k is a monotonically decreasing function with k and $\beta_k < 4$ for $K \ge 2$. As $k \to \infty$, β_k approaches e. Therefore for any covariance matrix $\boldsymbol{Q}$, a uniformly distributed random code converges to $\boldsymbol{Q}$ according to the $\|\cdot\|_F$ norm exponentially with the number of bits at a rate that is at least $\frac{1}{2k-2}$ for a fixed number $k = MM_t$. Also, the above bound presents a very intuitive result. In order to ensure the convergence of the randomly generated codebook when the number of antennas at the transmitter or receiver is increased, the number of feedback bits should be increased by a factor which is less than or equal to $\frac{1}{2k-2}$.

IV. Capacity Loss and Feedback Rate

In this section, we will derive expressions for the capacity loss associated with 1) a mismatch in the transmit covariance and 2) a covariance matrix designed using limited feedback.

A. Introduction to Capacity Loss

In general, the capacity loss depends on the amount of channel state information at the transmitter or, in other words, the number of feedback bits transmitted per channel realization. Theoretically the amount of feedback is infinite when a complete CSIT is assumed, and the amount of feedback is equal to zero when there is no CSI at the transmitter. As we have mentioned earlier in the latter case when the Gaussian channel model is considered, the ergodic capacity is maximized by i.i.d. M_t streams of transmitted data. The amount of capacity loss relative to the waterfilling performance due to this i.i.d. suboptimal strategy was considered before and bounds on the capacity loss where derived in [38], [39]. Using the results in [38], [40], it can be shown that (when $M_r \ge M_t$)

$$C_{\mathrm{IT}}(\rho;\boldsymbol{H}) - C_{\mathrm{UT}}(\rho;\boldsymbol{H}) \le M_t e^{-1}\log e \tag{20}$$

where C_{IT} and C_{UT} are given by (3) and (6), respectively. The above bound is a universal bound on the capacity loss for any SNR and channel realization. In [39], it was shown that in the high SNR regime (when $M_r \ge M_t$)

$$C_{\mathrm{IT}}(\rho;\boldsymbol{H}) - C_{\mathrm{UT}}(\rho;\boldsymbol{H}) \le M_t\log\left(1+\frac{\sigma_{min}^{-1}-\sigma_{\max}^{-1}}{\rho/M_t}\right) \tag{21}$$

where $\sigma_{\min}$ and $\sigma_{\max}$ are the minimal and maximal eigenvalues of the matrix $\boldsymbol{H}^*\boldsymbol{H}$, respectively. In this case the bound does depend on the SNR and channel matrix $\boldsymbol{H}$, and

$$\lim_{\rho\to\infty} C_{\mathrm{IT}}(\rho;\boldsymbol{H}) - C_{\mathrm{UT}}(\rho;\boldsymbol{H}) = 0. \tag{22}$$

Therefore, in the high SNR regime the optimal transmission strategy is that of i.i.d. streams of data transmitted along each of the M_t transmit antennas. In the following, we will derive a bound on the capacity loss for the more general case when an arbitrary covariance matrix is used instead of the optimal waterfilling matrix assuming no restrictions on the number of transmit and receive antennas.

We are interested in characterizing 1) the capacity loss in bits/s/Hz when $\widetilde{\boldsymbol{Q}}$ is used instead of $\boldsymbol{Q}_{IT}$ and 2) the average loss in bits/s/Hz incurred from using B bits of feedback and a random codebook $\mathcal{Q}$. The *capacity loss caused by a suboptimal covariance* assuming a fixed and deterministic channel $\boldsymbol{H}_k = \boldsymbol{H}$ is given by

$$\Delta C_{\widetilde{\boldsymbol{Q}}}(\rho; \boldsymbol{H}) \equiv C_{\mathrm{IT}}(\rho; \boldsymbol{H}) - C_{\widetilde{\boldsymbol{Q}}}(\rho; \boldsymbol{H}).$$

The *average capacity loss with feedback* when using a random codebook over an i.i.d. Rayleigh channel is given by

$$\Delta C(\rho) \equiv C_{\mathrm{IT}}(\rho) - E\left[\max_{\widetilde{\boldsymbol{Q}} \in \mathcal{Q}} C_{\widetilde{\boldsymbol{Q}}}(\rho; \boldsymbol{H}_k)\right]$$

where the expectation is over both $\boldsymbol{H}_k$ and $\mathcal{Q}$. Because of the linearity of the expectation operator, the average capacity loss can alternatively be written as

$$\Delta C(\rho) = E\left[\min_{\widetilde{\boldsymbol{Q}} \in \mathcal{Q}} \Delta C_{\widetilde{\boldsymbol{Q}}}(\rho; \boldsymbol{H}_k)\right]. \quad (23)$$

B. Capacity Loss Results

Before we state the main theorem on the ergodic capacity loss bound we will need the following lemma.

Lemma 4: Let $\boldsymbol{Q}_{IT}$ denote the normalized covariance matrix obtained from waterfilling and $\boldsymbol{\Delta Q} \equiv \boldsymbol{Q}_{IT} - \widetilde{\boldsymbol{Q}}$. The capacity loss is bounded as

$$\Delta C_{\widetilde{\boldsymbol{Q}}}(\rho; \boldsymbol{H}) \leq \frac{M}{2} \log\left(\frac{\left\|\boldsymbol{I}_M + \boldsymbol{J}^{-1}\boldsymbol{\delta J}\right\|_F^2}{M}\right) \quad (24)$$

where $\boldsymbol{J}$ and $\boldsymbol{\delta J}$ are defined as

$$\boldsymbol{J} \equiv \left(\boldsymbol{I}_{M_t} + \rho \boldsymbol{H}^* \boldsymbol{H} \widetilde{\boldsymbol{Q}}\right), \quad \boldsymbol{\delta J} \equiv \rho \boldsymbol{H}^* \boldsymbol{H} \boldsymbol{\Delta Q} \quad (25)$$

when $M_t \leq M_r$ and

$$\boldsymbol{J} \equiv \left(\boldsymbol{I}_{M_r} + \rho \boldsymbol{H} \widetilde{\boldsymbol{Q}} \boldsymbol{H}^*\right), \quad \boldsymbol{\delta J} \equiv \rho \boldsymbol{H} \boldsymbol{\Delta Q} \boldsymbol{H}^* \quad (26)$$

when $M_t > M_r$.

Proof: First note that in both cases ($M_t \leq M_r$ and $M_t > M_r$) the capacity loss can be written as

$$\Delta C_{\widetilde{\boldsymbol{Q}}}(\rho; \boldsymbol{H}) = \log\left(\frac{\det(\boldsymbol{J} + \boldsymbol{\delta J})}{\det(\boldsymbol{J})}\right) = \det\left(\boldsymbol{I}_M + \boldsymbol{J}^{-1}\boldsymbol{\delta J}\right). \quad (27)$$

Now according to the Hadamard inequality [41], for any $n \times n$ matrix $\boldsymbol{A}$, it's determinant is bounded as

$$\det(\boldsymbol{A}) \leq \prod_{j=1}^{n} \sqrt{\sum_{i=1}^{n} |a_{ij}|^2} \quad (28)$$

where a_{ij} is the element of $\boldsymbol{A}$ corresponding to the ith row and jth column of $\boldsymbol{A}$. Therefore,

$$\begin{aligned} \det(\boldsymbol{A}) &\leq \prod_{j=1}^{n} \sqrt{\sum_{i=1}^{n} |a_{ij}|^2} \\ &= \left(\prod_{j=1}^{n}\left(\sum_{i=1}^{n}|a_{ij}|^2\right)^{1/n}\right)^{n/2} \\ &\leq \left(\frac{\sum_{j=1}^{n}\sum_{i=1}^{n}|a_{ij}|^2}{n}\right)^{n/2} \\ &= \left(\frac{\|\boldsymbol{A}\|_F^2}{n}\right)^{n/2} \end{aligned} \quad (29)$$

where the inequality in (29) follows from the geometric-arithmetic mean bound. Finally, substituting $\left(\boldsymbol{I}_M + \boldsymbol{J}^{-1}\boldsymbol{\delta J}\right)$ for $\boldsymbol{A}$ in (29) we get (24). □

Let ξ be defined as

$$\xi \equiv \sqrt{2M} N \frac{\Gamma\left(\frac{2k-1}{2k-2}\right)\Gamma(2^B)}{\Gamma\left(\frac{2k-1}{2k-2} + 2^B\right)} \quad (30)$$

where this is merely the bound in (13) which is a function of B and the number of transmit and receive antennas. Also let

$$\gamma \equiv E_{\boldsymbol{H}_k}\left[\|\boldsymbol{H}_k\|_2^2\right] \quad (31)$$

where $\|\cdot\|_2$ is the matrix spectral norm. Then the following theorem follows.

Theorem 3: The ergodic capacity loss with B bits of feedback is bounded as

$$\Delta C(\rho) \leq M \log(1 + \rho\gamma\xi) \quad (32)$$

where γ and ξ are defined as in (30) and (31), respectively, and ρ is the signal-to-noise ratio.

Proof: Let $\sigma_i(\boldsymbol{A})$ denote the ith singular value of a matrix $\boldsymbol{A}$, then the bound in (24) can be written as

$$\frac{M}{2}\log\left(\frac{\left\|\boldsymbol{I}_M + \boldsymbol{J}^{-1}\boldsymbol{\delta J}\right\|_F^2}{M}\right) = \frac{M}{2}\log\left(\frac{\sum_{i=1}^{M}\sigma_i^2\left(\boldsymbol{I}_M + \boldsymbol{J}^{-1}\boldsymbol{\delta J}\right)}{M}\right). \quad (33)$$

From [41, p. 423], we have the following inequality. For any $n \times n$ matrix $\boldsymbol{A} = \boldsymbol{I}_n + \boldsymbol{B}$

$$\sigma_i(\boldsymbol{A}) \leq 1 + \sigma_i(\boldsymbol{B}). \quad (34)$$

Therefore, from (33) and (34) we get (35) shown at the bottom of the page. Now since

$$\left\|\boldsymbol{J}^{-1}\right\|_2 = \frac{1}{\sigma_M(\boldsymbol{J})} \leq 1 \quad (36)$$

$$\begin{aligned} \Delta C_{\widetilde{\boldsymbol{Q}}}(\rho; \boldsymbol{H}) &\leq \frac{M}{2}\log\left(\frac{\sum_{i=1}^{M}\sigma_i^2\left(\boldsymbol{I}_M + \boldsymbol{J}^{-1}\boldsymbol{\delta J}\right)}{M}\right) \\ &\leq \frac{M}{2}\log\left(\frac{\sum_{i=1}^{M}\left(1 + \sigma_i^2\left(\boldsymbol{J}^{-1}\boldsymbol{\delta J}\right) + 2\sigma_i\left(\boldsymbol{J}^{-1}\boldsymbol{\delta J}\right)\right)}{M}\right) \\ &\leq \frac{M}{2}\log\left(\frac{\sum_{i=1}^{M}\left(1 + \left\|\boldsymbol{J}^{-1}\right\|_2^2\|\boldsymbol{\delta J}\|_2^2 + 2\left\|\boldsymbol{J}^{-1}\right\|_2\|\boldsymbol{\delta J}\|_2\right)}{M}\right). \end{aligned} \quad (35)$$

where $\sigma_M(\boldsymbol{J})$ is the minimum singular value of $\boldsymbol{J}$

$$\begin{aligned}\Delta C_{\widetilde{\boldsymbol{Q}}}(\rho;\boldsymbol{H}) &\le \frac{M}{2}\log\left(\frac{\sum_{i=1}^{M}\left(1+\|\boldsymbol{\delta J}\|_2\right)^2}{M}\right)\\ &= M\log\left(1+\|\boldsymbol{\delta J}\|_2\right). \end{aligned}\tag{37}$$

Furthermore, when $M_t \le M_r$ or $M_t > M_r$

$$\|\boldsymbol{\delta J}\|_2 \le \rho\,\|\boldsymbol{H}\|_2^2\,\|\boldsymbol{\Delta Q}\|_2 \le \rho\,\|\boldsymbol{H}\|_2^2\,\|\boldsymbol{\Delta Q}\|_F. \tag{38}$$

Taking the expectation of (37) as in (23) gives (39)–(42) shown at the bottom of the page, where (39) follows from Jensen's inequality for a concave function, (41) follows from (19), and (42) uses Jensen's inequality again. □

Corollary 2: The ergodic capacity loss is bounded as

$$\Delta C(\rho) \le M\log\left(1+\beta_k\sqrt{2M}\rho M_t M_r 2^{-\frac{B}{2MM_t-2}}\right) \tag{43}$$

where

$$\beta_k \equiv \Gamma\left(\frac{2k-1}{2k-2}\right)\exp\left(\frac{2k-1}{2k-2}\right).$$

This corollary follows from Theorem 3, the bound in (19), and by noting that $E_{\boldsymbol{H}_k}\left[\|\boldsymbol{H}_k\|_2^2\right]$ can be bounded as

$$E_{\boldsymbol{H}_k}\left[\|\boldsymbol{H}_k\|_2^2\right] \le E_{\boldsymbol{H}_k}\left[\|\boldsymbol{H}_k\|_F^2\right] = M_t M_r$$

where the equality follows since $\boldsymbol{H}_k$ consists of $M_t M_r$ i.i.d. elements with zero mean and unit variance. □

Note that according to (43) when the system has a large feedback capacity where $B \gg M_t$ or when the signal-to-noise ratio is small

$$\Delta C(\rho) \le \beta_k\sqrt{2M}\log(e)\rho M M_t M_r 2^{-\frac{B}{2MM_t-2}}.$$

Therefore, the capacity loss decreases to zero exponentially with the number of bits for any value of SNR and number of transmit and receive antennas.

Let us recall that according to our assumption, the covariance matrix $\widetilde{\boldsymbol{Q}}$ is chosen from a randomly generated codebook such that it maximizes the expression of the mutual information in (1). Thus, for each channel use a codebook $\mathcal{Q} = \{\boldsymbol{Q}_1, \boldsymbol{Q}_2, \ldots, \boldsymbol{Q}_N\}$ is randomly generated where it is revealed to both the transmitter and the receiver (at zero cost), and the argument which maximizes the mutual information of the channel is transmitted over the feedback link. In general however, we are interested in a uniformly distributed random codebooks on the complex unit sphere. As mentioned earlier, the main reason for this is the optimality in a MISO system employing a covariance adaptation technique [13] and the robustness in a situation where the channel has rapidly changing statistics. However, another important reason for considering such uniform complex spherical distributions is the computational simplicity. If we take a look at (11), the cumulative distribution function (cdf) of the random variable Ψ is independent of the arbitrary chosen vector $\boldsymbol{v}$, which implies that the mean can be taken over the randomly generated codebook and the result will be independent of the channel realization.

If we look more carefully at the exponential term in the capacity loss bound, we can easily conclude that given a fixed bit rate B, the number of transmit antennas is more crucial than the number of receive antennas in terms of the convergence rate. As we can see, increasing the number of receive antennas will have an effect on the convergence rate only when M_r is less or equal to M_t. Because the number of transmit antennas always have the same degradation effect on the convergence rate, an increased number of bits is needed for any addition in the number of transmit antennas in order to keep the convergence rate fixed. In general, this is true because waterfilling is only a function of the M channel singular values. When $M_r \le M_t$, the number of receive antennas factors into reducing the dimensionality of the quantization problem. However, when $M_r \ge M_t$, the dimensionality of the problem is entirely specified by M_t.

From Corollary 1 of Theorem 1, we know that every covariance matrix in the set $\{\boldsymbol{Q}_i \in \mathcal{T}_M,\ 1 \le i \le N\}$ can be written as

$$\boldsymbol{Q}_i = \frac{\boldsymbol{G}_i^*\boldsymbol{G}_i}{\|\boldsymbol{G}_i\|_F^2} \tag{44}$$

where $\boldsymbol{G}_i$ is an $M \times M_t$ complex Gaussian matrix. Also, let κ_i denote the condition number of the matrix $\boldsymbol{G}_i$ and

$$\widetilde{\kappa} \equiv \min_{i=1}^{N} \kappa_i. \tag{45}$$

Then we have the following theorem.

Theorem 4: Let $\widetilde{\kappa}$ be the minimal condition number as defined in (45) and assume that $M_r \ge M_t$. In the high SNR regime, the ergodic capacity loss is bounded as

$$\Delta C(\rho) \le 2(M_t-1)E\left[\log\widetilde{\kappa}\right] \tag{46}$$

when $\rho \to \infty$.

$$\Delta C(\rho) = E_{\boldsymbol{H}_k}\left[E_{\mathcal{Q}}\left[\min_{\widetilde{\boldsymbol{Q}}\in\mathcal{Q}}\Delta C_{\widetilde{\boldsymbol{Q}}}(\rho;\boldsymbol{H}) \mid \boldsymbol{H}_k=\boldsymbol{H}\right]\right]$$

$$\le ME_{\boldsymbol{H}_k}\left[E_{\mathcal{Q}}\left[\min_{\widetilde{\boldsymbol{Q}}\in\mathcal{Q}}\log\left(1+\rho\,\|\boldsymbol{H}\|_2^2\,\|\boldsymbol{\Delta Q}\|_F\right) \mid \boldsymbol{H}_k=\boldsymbol{H}\right]\right]$$

$$\le ME_{\boldsymbol{H}_k}\left[\log\left(1+\rho E_{\mathcal{Q}}\left[\min_{\widetilde{\boldsymbol{Q}}\in\mathcal{Q}}\|\boldsymbol{H}\|_2^2\,\|\boldsymbol{\Delta Q}\|_F \mid \boldsymbol{H}_k=\boldsymbol{H}\right]\right)\right] \tag{39}$$

$$= ME_{\boldsymbol{H}_k}\left[\log\left(1+\rho\,\|\boldsymbol{H}_k\|_2^2\,E_{\mathcal{Q}}\left[\min_{\widetilde{\boldsymbol{Q}}\in\mathcal{Q}}\|\boldsymbol{\Delta Q}\|_F \mid \boldsymbol{H}_k=\boldsymbol{H}\right]\right)\right] \tag{40}$$

$$\le ME_{\boldsymbol{H}_k}\left[\log\left(1+\rho\,\|\boldsymbol{H}_k\|_2^2\,\xi\right)\right] \tag{41}$$

$$\le M\log\left(1+\rho E_{\boldsymbol{H}_k}\left[\|\boldsymbol{H}_k\|_2^2\right]\xi\right) \tag{42}$$

Proof: According to (22), in the high SNR regime the informed and uninformed transmitter capacities coincide. Therefore, the optimal waterfilling capacity in this case can be approximated by

$$C_{\mathrm{IT}}(\rho;\boldsymbol{H}) \approx \log\det\left(\boldsymbol{I}_{M_t} + \frac{\rho}{M_t}\boldsymbol{H}^*\boldsymbol{H}\right) \tag{47}$$

and the capacity loss in this case is given by

$$\begin{aligned}\Delta C_{\boldsymbol{Q}_i}(\rho;\boldsymbol{H}) &\approx \log\det\left(\boldsymbol{I}_{M_t} + \frac{\rho}{M_t}\boldsymbol{H}^*\boldsymbol{H}\right)\\ &\quad - \log\det\left(\boldsymbol{I}_{M_t} + \rho\boldsymbol{H}^*\boldsymbol{H}\boldsymbol{Q}_i\right)\\ &\approx \log\det\left(\frac{\rho}{M_t}\boldsymbol{H}^*\boldsymbol{H}\right) - \log\det\left(\rho\boldsymbol{H}^*\boldsymbol{H}\boldsymbol{Q}_i\right)\end{aligned}$$

where $\boldsymbol{Q}_i \in \mathcal{T}_M$ is an arbitrary covariance matrix generated from a corresponding $M \times M_t$ matrix $\boldsymbol{G}_i$ according to (44). Now assuming that the matrix $\boldsymbol{H}^*\boldsymbol{H}$ is full rank (i.e., $M_r \geq M_t$) then,

$$\Delta C_{\boldsymbol{Q}_i}(\rho;\boldsymbol{H})_{\rho\to\infty} = M_t \log\left(\frac{1}{M_t}\right) - \log\det\left(\boldsymbol{Q}_i\right). \tag{48}$$

The determinant of $\boldsymbol{Q}_i$ can be lower bounded as follows:

$$\det\left(\boldsymbol{Q}_i\right) = \left|\det\left(\frac{\boldsymbol{G}_i}{\|\boldsymbol{G}_i\|_F}\right)\right|^2 = \left|\frac{\det\left(\boldsymbol{G}_i\right)}{\|\boldsymbol{G}_i\|_F^{M_t}}\right|^2$$

and from the fact that $\|\boldsymbol{G}_i\|_F^2 = \sum_{i=1}^{M_t}\sigma_i^2$, where $\sigma_i, 1 \leq i \leq M_t$, are the singular values of $\boldsymbol{G}_i$

$$\begin{aligned}\det\left(\boldsymbol{Q}_i\right) &\geq \frac{|\det\left(\boldsymbol{G}_i\right)|^2}{M_t^{M_t}\sigma_1^{2M_t}} = \frac{\sigma_1^{-2(M_t-1)}\sigma_2^2\cdots\sigma_{M_t}^2}{{M_t}^{M_t}}\\ &\geq \frac{\sigma_1^{-2(M_t-1)}\sigma_{M_t}^{2(M_t-1)}}{{M_t}^{M_t}}\\ &= \frac{1}{M_t^{M_t}}\left(\frac{\sigma_{M_t}}{\sigma_1}\right)^{2(M_t-1)}.\end{aligned} \tag{49}$$

Therefore, in the high SNR regime

$$\begin{aligned}\Delta C_{\boldsymbol{Q}_i}(\rho;\boldsymbol{H})_{\rho\to\infty} &\leq M_t\log\left(\frac{1}{M_t}\right)\\ &\quad - \log\left(\frac{1}{M_t^{M_t}}\left(\frac{1}{\kappa_i}\right)^{2(M_t-1)}\right)\\ &= 2(M_t-1)\log(\kappa_i).\end{aligned}$$

Finally, (46) follows from the fact that log is a monotonic function and the minimal capacity loss taken over the different codebook elements is upper bounded by

$$2(M_t-1)\min_{i=1}^{N}\log(\kappa_i).$$

□

As we can see, in the high SNR regime the bound does not depend on ρ or the channel realization, but it does depend on M_t and the minimal condition number of the set of matrices $\{\boldsymbol{G}_i,\ 1 \leq i \leq N\}$. Also, when $\log\widetilde{\kappa}$ is equal to zero the capacity loss is equal to zero. In this case at least one of the matrices $\{\boldsymbol{G}_i,\ 1 \leq i \leq N\}$ (up to a scaling factor) must be a unitary matrix which is the case when the identity matrix is included in the codebook. In general, for an arbitrary number of transmit and receive antennas, a simple closed-form expression for the expectation in (46) is difficult to achieve. However, in the following theorem we give the result for any system employing $2 \times M_r$ antennas.

Theorem 5: For a system employing $2 \times M_r$ antennas where $M_r \geq 2$, the ergodic capacity loss in the high SNR regime is bounded as

$$\begin{aligned}\Delta C(\rho) \leq 2^N\log(e)\sum_{k=0}^{N}\binom{N}{k}3^k\\ \times\frac{{}_2F_1(3N, 3N-2k; 3N-2k+1; -1)}{3N-2k}\end{aligned}$$

where N is the size of the randomly generated codebook with uniform distribution on the complex unit sphere and ${}_2F_1(\cdot,\cdot;\cdot;\cdot)$ is the Gaussian Hypergeometric function.

Proof: Based on the results in [42], the probability density function of the condition number of a randomly generated complex Gaussian matrix with $2 \times L$ i.i.d. elements is given by

$$f(x) = 2\frac{\Gamma(2L)}{\Gamma(L)\Gamma(L-1)}\frac{x^{2L-3}(x^2-1)^2}{(x^2+1)^{2L}}. \tag{50}$$

The probability density function of $\widetilde{\kappa}$ the minimal condition number of N complex Gaussian random matrices is therefore given by

$$f_{\widetilde{\kappa}}(x) = N(1-F(x))^{N-1}f(x)$$

where $F(x)$ is the cumulative distribution function of the condition number of a random complex Gaussian matrix. From integration by parts we get

$$\begin{aligned}E\left[\log\widetilde{\kappa}\right] &= \log(e)\int_1^\infty \ln(x)f_{\widetilde{\kappa}}(x)dx\\ &= \log(e)\int_1^\infty\frac{(1-F(x))^N}{x}dx.\end{aligned}$$

For the case when $M_t = 2$, then by (50)

$$1-F(x) = \frac{2+6x^4}{(1+x^2)^3}$$

and

$$\begin{aligned}E\left[\log\widetilde{\kappa}\right] &= \log(e)\int_1^\infty\frac{(2+6x^4)^N}{x(1+x^2)^{3N}}dx\\ &= 2^N\log(e)\sum_{k=0}^{N}\binom{N}{k}3^k\\ &\quad\times\int_1^\infty\frac{x^{4k-1}}{(1+x^2)^{3N}}dx.\end{aligned} \tag{51}$$

Now after a change of variable

$$\int_1^\infty\frac{x^{4k-1}}{(1+x^2)^{3N}}dx = \frac{1}{2}\int_0^\infty\frac{(1+u)^{2k-1}}{(2+u)^{3N}}du$$

and by [37, p. 315, eq. 3.197.9], we finally get

$$\begin{aligned}E\left[\log\widetilde{\kappa}\right] = 2^{N-1}\log(e)\sum_{k=0}^{N}\binom{N}{k}3^k\\ \times\frac{{}_2F_1(3N; 3N-2k; 3N-2k+1; -1)}{3N-2k}.\end{aligned} \tag{52}$$

□

According to this result, the ergodic capacity loss bound is a function of the codebook cardinality only, and by simulation results it will be later shown how this capacity loss scales exponentially with the number of feedback bits used to generate the codebook. As mentioned before, since in the high SNR regime the informed and uninformed transmitter capacities coincide, and from the waterfilling algorithm [1], [30], [39],

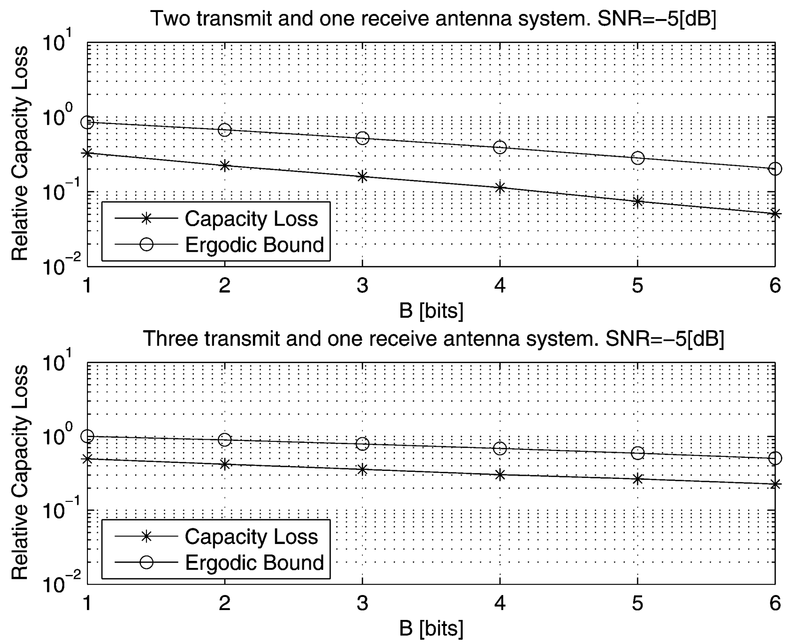

Fig. 1. Numerically simulated scaled capacity loss and the analytically derived bound on the ergodic capacity loss for two different antenna system configurations operating at an SNR of -5 [dB]. Note that the numerical simulation displays an exponential decrease in B.

[43] we know that asymptotically the transmitted power on the different eigenmodes of the channel are equal, the optimal waterfilling covariance matrix $\boldsymbol{Q}_{\rm IT}$ will converge to $\frac{1}{M_t}\boldsymbol{I}_{M_t}$ as $\rho \rightarrow \infty$ for all full rank channels when $M_r \geq M_t$. Hence, in order to reduce the capacity loss especially in the high SNR regime, the covariance matrix $\frac{1}{M_t}\boldsymbol{I}_{M_t}$ should be included in the codebook $\mathcal{Q}$. Including $\frac{1}{M_t}\boldsymbol{I}_{M_t}$ allows 1) the robust signaling bound in (20) to be used to obtain an upper bound on the ergodic capacity loss and 2) means that the codebook will be asymptotically optimal.bb

V. Discussion

In this section, we will address several issues relating to the above analysis.

A. Feedback Overhead Calculations

One possible application of the capacity loss-feedback rate tradeoff analysis is to the design of feedback rates in two-way communication. This kind of analysis was recently initiated in [44], [45] where $C_{\rm IT}(\rho) - C_{\rm UT}(\rho)$ determines what benefits feedback could provide over open-loop transmission.

Following the model in [44], [45], consider an $M_t = M_r = M$ two-way symmetric MIMO model where information is transmitted at the same rate on each side of the link. In addition, assume that the forward and reverse links fade independently and that feedback must be conveyed every T_c seconds. This necessitates a feedback rate of at least B/T_c bits/second. Therefore, for feedback to be beneficial, we must minimize

$$\Delta C(\rho) - \frac{B}{\Theta T_c}$$

where Θ is the bandwidth of the system. The factor ΘT_c can be thought of as a feedback penalty term. The feedback rate will be highly dependent on ΘT_c.

B. Application to Other Channel Models

In the derivations we made in the last section, we assumed that the channel matrix consists of $M_t M_r$ i.i.d. entries. However, this is not necessary. The ergodic capacity loss bound given in Corollary 2 is also true for any channel matrix with entries having variance less than or equal to one. The only assumption we needed in the above proof was that the variance is less than or equal to one for all entries since

$$E\left[\|\boldsymbol{H}_k\|_F^2\right] = \sum_{i=1}^{M_r}\sum_{j=1}^{M_t} E\left[\left|h_{i,j}^k\right|^2\right] \leq M_t M_r$$

if $E\left[\left|h_{i,j}^k\right|^2\right] \leq 1$. Hence, our result can be easily generalized for any channel with any distribution including correlated fading channels. If we normalize the channel matrix $\boldsymbol{H}$ such that all of its entries having variance less or equal to one, we can always use the bound derived and the result remain true. Furthermore, the high SNR ergodic capacity loss bound given in Theorem 4 is always true if the channel matrix is full rank (this includes correlated channels with full rank) with $M_r \geq M_t$.

VI. Simulation Results

In this section, we consider the flat Rayleigh fading MIMO channel. The channel matrix $\boldsymbol{H}_k$ in this case consists of i.i.d. complex Gaussian entries $h_{i,j}$ with zero mean and unit variance. For this type of channel, we compute the capacity loss incurred by the uniform spherically distributed random codebook which is generated according to the steps given in Section III.

Capacity Loss versus Feedback Rate: Theorem 3 showed that the loss in capacity decreases exponentially with the number of feedback bits. Figs. 1 and 2 demonstrate this scaling result for two different antenna system configurations operating at SNR values -5 and 0 dB, respectively. In both figures, we consider 2×1 and 3×1 antenna systems. Both numerical simulations and the derived bound in Theorem 3 are presented. Note that the demonstrated results are scaled by the optimal waterfilling capacity. First, from the figures we note that the capacity loss decreases exponentially with the number of feedback bits, and as expected from Theorem 3, the scaling in the capacity loss strongly depends on the number of transmit antennas. We also note that in general the ergodic capacity loss bound is fairly tight, and since the slopes of the two curves are similar we can conclude that the derived bound estimates the exponential decay rate as a function of the number of bits.

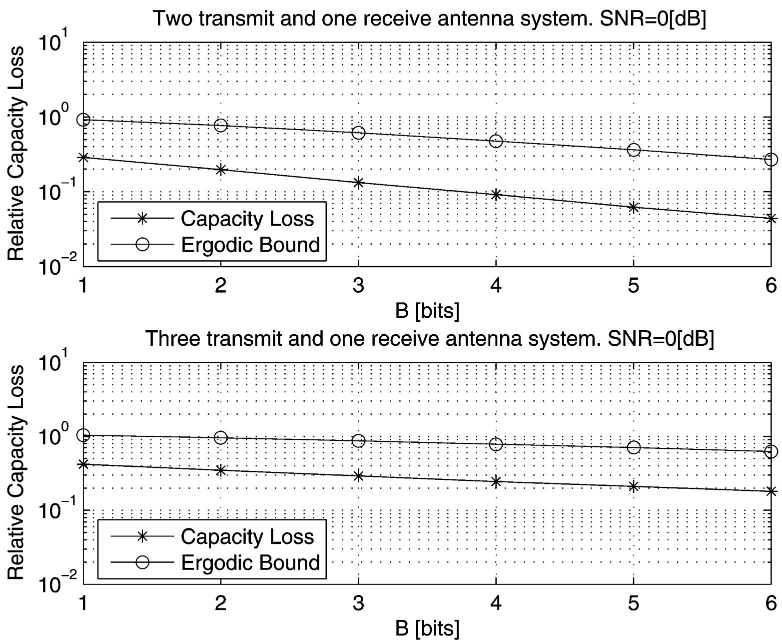

Fig. 2. Numerically simulated scaled capacity loss and the analytically derived bound on the ergodic capacity loss for two different antenna system configurations operating at an SNR of 0 [dB]. Note that the numerical simulation displays an exponential decrease in B.

Also, by comparing Figs. 1 and 2 the ergodic capacity loss bound tends to be less tight as the SNR is increased. One reason for this behavior can be explained by investigating the capacity loss bound in (43), where it can be easily shown that when the SNR is large and the communication system employs a low resolution feedback link, i.e., the number of feedback bits is of the order of MM_t, the ergodic capacity loss bound decays only linearly with the number of feedback bits. It should be also noted here that in the special case of $M_r = 1$, a tighter upper bound in the exponent (i.e., larger decay rate as a function of the number of feedback bits) of (43) was found in [46].

Random versus Lloyd Codebooks: Prior work [3], [4] has considered vector quantization techniques for generating covariance codebooks. Given the size of the codebook N, we generate $N-1$ random covariance matrices from the uniform distribution on the complex unit sphere, and we let the identity matrix $\frac{1}{M_t}\boldsymbol{I}_{M_t} \in \mathcal{T}_{M_t}$ be the Nth element in the codebook. The reason for the inclusion of the identity matrix has been mentioned in Section IV, and it should improve the performance of the randomly generated codebook especially in the high SNR regime. For this type of channel and random codebook we compute the average capacity as a function of the SNR using a Monte Carlo simulation. We also compare with the average capacity obtained using the deterministic codebooks designed by the vector quantization Lloyd algorithm as proposed in [3], [4].

We first consider a two transmit and two receive antenna system using codebooks with cardinality $B = 1, 2, 3,$ and 4 bits. The results of the average capacity loss for both quantized covariance feedback transmission schemes are shown in Fig. 3. Interestingly, as we can see from Fig. 3, for SNR values which are approximately above 5 dB, the uniform spherically distributed random coding method, achieves a better performance compared to the Lloyd based codebook design. This result is true for all the sizes of the codebooks used in the simulations, and the superiority of the random coding method becomes more evident as the size of the codebook increases. Note that the random codebook technique was simulated to include $\frac{1}{M_t}\boldsymbol{I}_{M_t} \in \mathcal{Q}$. According to (22), this is supposed to give asymptotic performance improvements. Our results show that the inclusion of the identity matrix in the uniform spherically distributed random codebook as mentioned above, is the cause for such superior performance. The random codebook performance benefits would be greatly reduced if an optimal, but computationally complex, Lloyd codebook design was conducted. We optimized the Lloyd designed codebook over 10 000 channel realizations beginning from ten different initial codebooks. For SNR values below 5 dB, the Lloyd feedback scheme achieves a higher system capacity.

Capacity Loss versus Feedback Rate in the High SNR Regime: As mentioned above, when the SNR is large and the communication system employs a low resolution feedback link, the ergodic capacity loss bound decays only linearly with the number of feedback bits. Therefore, it is expected that in this case the ergodic capacity loss bound given in (32) will be overbounded for reasonable values of B. In order to remedy this problem we can use the derived bound in Theorem 5. In this case the bound is only a function of the number of feedback bits used to generated the different codebook elements. In Fig. 4, the high SNR ergodic capacity loss bound given in Theorem 5 is plotted for a two transmit and two receive antenna system operating at an SNR of 20 dB. The bound is plotted versus the number of feedback bits. On the same figure, the true capacity loss and the ergodic capacity loss bound from Theorem 3 are also plotted, and for comparison the uninformed transmitter capacity loss bound given in (21) is also plotted on the same figure.

As we can see from the figure, the high SNR bound is tighter, and it demonstrates an exponential decrease with the number of feedback bits. A similar simulation was performed for a system employing three transmit and three receive antennas and the results are shown in Fig. 5. From Fig. 4 we can see that for $B \geq 8$ the ergodic capacity loss bound is less than the uninformed transmitter capacity loss bound. From the above figures it is also clear that the ergodic capacity loss bound is overbounded for this range of feedback bits, whereas the high SNR bound is relatively tight and attains an exponential decrease with the number of feedback bits. Finally, in the high SNR regime we can improve on the capacity loss by letting $\frac{1}{M_t}\boldsymbol{I}_{M_t}$ be one of the elements in the codebook and generating the $N-1$ other covariance matrices randomly as before.

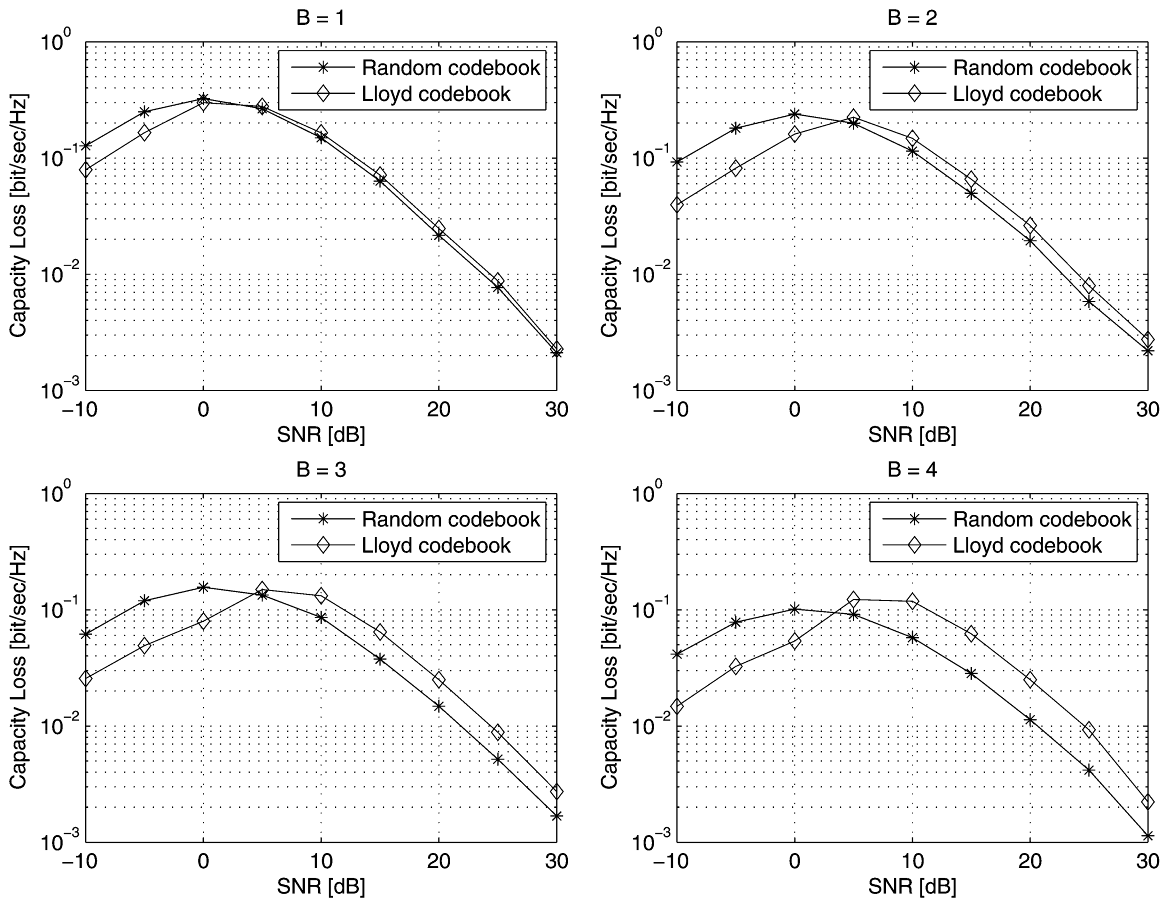

Fig. 3. Uniform distribution complex unit sphere random coding versus Lloyd quantized codebook for a two transmit and two receive antenna system.

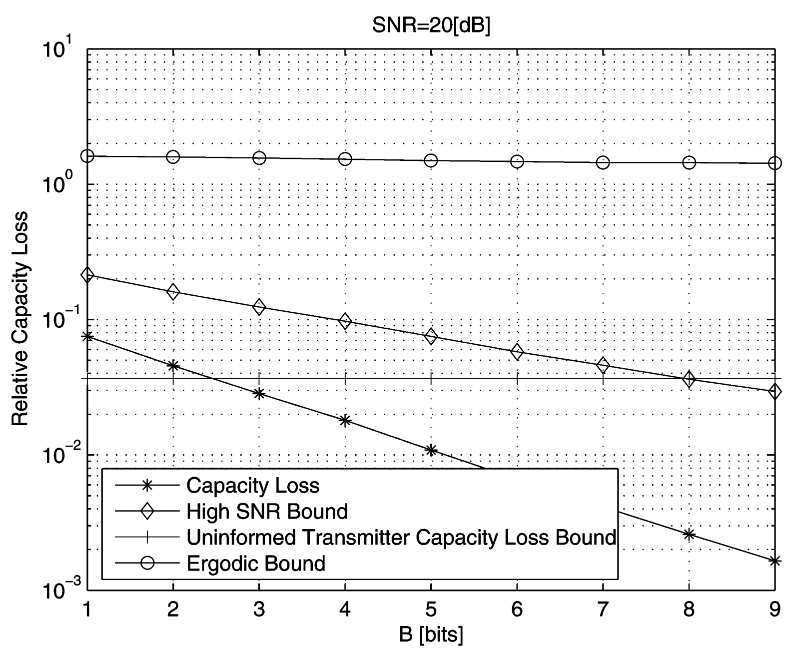

Fig. 4. Numerically simulated scaled capacity loss and the analytically derived high SNR bound for a two transmit and two receive antenna system operating at an SNR of 20 dB.

VII. Conclusion

In this paper, we considered the MIMO channel with quantized channel state information. For any covariance matrix with power constraint equal to one, we found that it can be generated from a corresponding vector on the complex unit sphere. Given an arbitrary covariance matrix, we bound the mean of the covariance error norm, when a uniform spherically distributed random codebook is considered, by finding the probability density function for the chordal

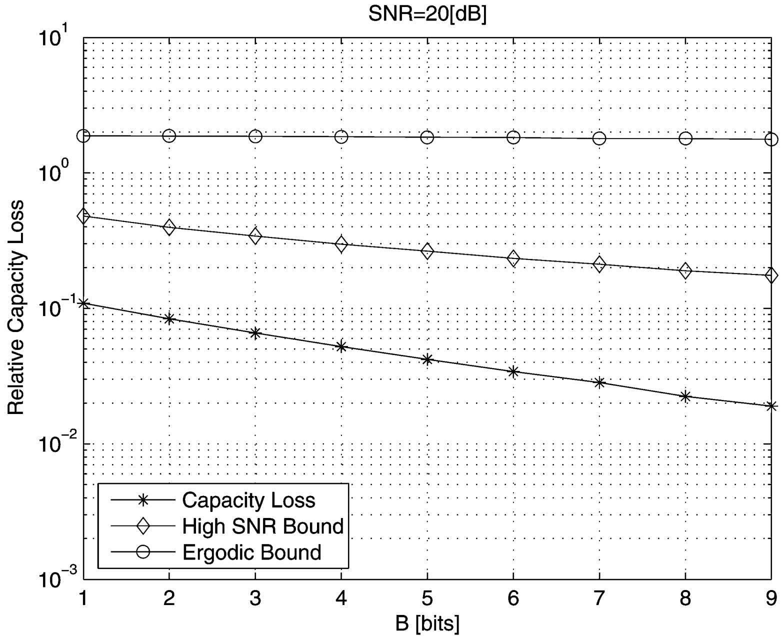

Fig. 5. Numerically simulated scaled capacity loss and the analytically derived high SNR bound for a three transmit and three receive antenna system operating at an SNR of 20 dB.

distance between an arbitrary complex unit sphere vector and a uniform randomly generated vector on the complex unit sphere.

For a randomly generated codebook with uniform distribution on the complex unit sphere we derived a bound on the ergodic capacity loss as a function of the number of feedback bits. We have also shown that this bound decreases as $O\left(2^{-B/(2MM_t-2)}\right)$ for B bits of feedback. For a communication system operating in the high SNR regime with a low resolution feedback link, we derived a bound which is a function of the minimal condition number of N randomly generated Gaussian matrices. In this case, we also derived a closed-form expression for the ergodic capacity loss bound in a system with two transmit antennas and any number of receive antennas which is greater than or equal to two.

In the case of flat-fading Rayleigh MIMO channel, it was found according to simulation results that for SNR values above 5 dB, a uniform spherically distributed random codebook which includes the identity matrix is comparable in terms of the average capacity loss to a Lloyd deterministic codebook designed under practical design methods.

One piece of future analysis that would be interesting is to consider the time-varying power allocation strategy using the long-term power constraint discussed in [47]. In this correspondence, power is not allowed to be allocated in time to adjust to channel conditions. Removing this restriction could substantially improve performance. It is unclear how the capacity loss scaling as a function of the feedback amount would be affected. It is also of interest to understand the exact behavior of the asymptotic slope of the capacity loss as a function of B. Our analysis provides a bound on this slope, but this bound can be improved upon in the case of a single receive antenna [46].

This work actually opens up a new area of information theory that bridges traditional rate-distortion theory and channel capacity theory. We were able to generate a rate-distortion-like curve where rate refers to *feedback rate* and distortion refers to *capacity loss.* As limited rate feedback becomes important to industry, as currently being witnessed in the IEEE 802.16E and IEEE 802.11N standards bodies, it will be imperative to understand if the benefits of feedback outweigh the associated problems. One exciting area of future work is to analyze the feedback benefits when the transmitter already has some form of statistical or partial channel knowledge at the transmitter (see for example [48], [49]). A large number of open problems remain.

Appendix
Proof of Theorem 1

We will prove this theorem in several steps.

Lemma 5: Every matrix in $\mathcal{T}_M$ can be written as $\boldsymbol{F}^*\boldsymbol{F}$, where $\mathrm{vec}(\boldsymbol{F})$ is an MM_t-dimensional unit vector. In addition, each vector $\boldsymbol{w} \in \Omega_{MM_t}$ generates a matrix $\boldsymbol{Q} \in \mathcal{T}_M$ where $\boldsymbol{Q} = \mathrm{unvec}(\boldsymbol{w})^*\mathrm{unvec}(\boldsymbol{w})$.

Proof: Each matrix $\boldsymbol{Q}$ in $\mathcal{T}_M$ can be defined by its eigenvalue decomposition

$$\boldsymbol{Q} = \boldsymbol{U}\boldsymbol{\Sigma}^2\boldsymbol{U}^*$$

where $\boldsymbol{\Sigma}^2$ has only M possible nonzero diagonal entries, $\boldsymbol{\Sigma}$ is the corresponding $M \times M_t$ diagonal matrix with entries which are the square roots of the nonzero eigenvalues, and $\boldsymbol{U}$ is a $M_t \times M_t$ unitary matrix. Let $\boldsymbol{F} = \boldsymbol{\Sigma}\boldsymbol{U}^*$. By the power constraint we must have that $\mathrm{tr}(\boldsymbol{F}^*\boldsymbol{F}) = \mathrm{vec}(\boldsymbol{F})^*\mathrm{vec}(\boldsymbol{F}) = 1$. From this analysis, it is obvious that for any $\boldsymbol{w} \in \Omega_{MM_t}$ we have that $\mathrm{tr}\left(\mathrm{unvec}(\boldsymbol{w})^*\mathrm{unvec}(\boldsymbol{w})\right) = 1$ and thus $\mathrm{unvec}(\boldsymbol{w})^*\mathrm{unvec}(\boldsymbol{w}) \in \mathcal{T}_M$. □

Now we prove Theorem 1.

Proof: From Lemma 5, it is obvious that if we left multiply the matrix $\boldsymbol{F} = \boldsymbol{\Sigma}\boldsymbol{U}^*$ by any unitary matrix $\boldsymbol{V} \in U(M)$, we still get the same result, i.e.

$$\boldsymbol{Q} = (\boldsymbol{V}\boldsymbol{F})^*(\boldsymbol{V}\boldsymbol{F}) = \boldsymbol{F}^*\boldsymbol{V}^*\boldsymbol{V}\boldsymbol{F} = \boldsymbol{F}^*\boldsymbol{F}.$$

Define an equivalence relation on Ω_{MM_t} such that $\boldsymbol{w}_1$ is equivalent to $\boldsymbol{w}_2$ if

$$\mathrm{unvec}(\boldsymbol{w}_1)^*\mathrm{unvec}(\boldsymbol{w}_1) = \mathrm{unvec}(\boldsymbol{w}_2)^*\mathrm{unvec}(\boldsymbol{w}_2).$$

It follows from this that $\mathcal{T}_M$ is equal to the quotient space $\Omega_{MM_t}/U(M)$. Let $\boldsymbol{w}$ denote an arbitrary vector in Ω_{MM_t} and $\boldsymbol{W} = \mathrm{unvec}(\boldsymbol{w})$. For the one-to-one mapping, let the singular-value decomposition of any matrix $\boldsymbol{W} = \mathrm{unvec}(\boldsymbol{w})$ with $\boldsymbol{w} \in \Omega_{MM_t}$ be denoted as $W = V\Sigma U^*$. The singular value decomposition by itself is not unique, but the matrix $F = \Sigma U^*$ (or equivalently the matrix $Q = F^* F$) is unique given a left singular matrix realization V. Therefore we can define a unique mapping from Ω_{MM_t} to $(\mathcal{T}_M, U(M))$. □

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