Diversity Performance of Precoded Orthogonal Space-Time Block Codes Using Limited Feedback

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***Abstract*—Orthogonal space-time block codes (OSTBCs) are simple space-time codes that can be used for open-loop transmit diversity systems. OSTBCs, however, can only be designed for certain numbers of transmit antennas. Channel-dependent linear precoders have been proposed to overcome this deficiency, but it is not clear what conditions the precoder design must satisfy to guarantee full diversity order. In this letter, we show necessary and sufficient conditions for linear precoded OSTBCs to provide full diversity order. We show that limited feedback precoding can achieve full diversity order using fewer bits than limited feedback beamforming. We also present a simplified version of antenna subset selection for OSTBCs that can provide full diversity order with low complexity and only a small amount of feedback.**

***Index Terms*—Antenna selection, diversity methods, feedback, MIMO systems, space-time coding.**

I. Introduction

ORTHOGONAL space-time block codes (OSTBCs), first proposed in [1], [2], allow wireless systems to exploit spatial diversity from multiple transmit antennas. Unfortunately, OSTBCs can not be designed for arbitrary numbers of transmit antennas with complex-field modulation schemes [2]. For this reason, modified OSTBCs that can adapt to the channel based on current channel conditions have been studied in [3]–[9]. When channel knowledge is not available at the transmitter, OSTBCs can be adapted to channel conditions using a low rate feedback path from the receiver to the transmitter that can carry information about the channel [4]–[7]. This limited feedback channel information might represent a quantized channel estimate [7], an antenna subset [5], [6], or a matrix index in a finite set (or codebook) of precoder matrices [4]. All three limited feedback options can be viewed as sending back a limited number of bits representing a matrix index in a codebook of possible precoder matrices known *a priori* to both the transmitter and receiver.

Diversity order, defined to be the negative of the asymptotic slope of the probability of error curve, is an important indicator of the performance of any multi-antenna signaling method. Because it is essential for precoded OSTBCs to maintain the full diversity order, it is important to find out what conditions the codebook of precoding matrices must satisfy in order to guarantee that diversity order is maximized. This letter addresses this diversity analysis problem and finds necessary and sufficient conditions for full diversity order on the design of limited feedback precoders for OSTBCs. We also compare the minimum feedback requirements of precoded OSTBCs with beamforming and subsequently propose a new method of reduced complexity antenna subset selection.

II. Precoded OSTBCs

Consider an M_t transmit and M_r receive antenna wireless system transmitting OSTBCs. OSTBCs work by mapping a block of n_s symbols $s_1, s_2, \ldots, s_{n_s}$ taken from a constellation $\mathcal{S}$ into a matrix codeword. For example, an M antenna OSTBC produces an $M \times T$ ($M \leq M_t$) space-time code matrix $\mathbf{C}(k) = [\mathbf{c}_1(k)\ \mathbf{c}_2(k)\ \ldots\ \mathbf{c}_T(k)]$ at the kth codeword transmission. To constrain transmit power, we will assume that[1] $E_{s_l}\left[|s_l|^2\right] = 1$.

In a precoded OSTBC, the codeword matrix is multiplied by an $M_t \times M$ matrix $\mathbf{F}$ and sent over M_t transmit antennas. The channel $\mathbf{H}$ between the M_t transmit antennas and M_r receive antennas is modeled as

$$\mathbf{H} = \mathbf{R}_R^{1/2}\mathbf{H}_w\mathbf{R}_T^{1/2} \tag{1}$$

where $\mathbf{H}_w$ is a random matrix with independent entries distributed according to $\mathcal{CN}(0,1)$, $\mathbf{R}_R$ is the receive correlation matrix, and $\mathbf{R}_T$ is the transmit correlation matrix. This model was first developed in [10], experimentally vaidated in [11], and proposed for use by the IEEE 802.11N task group in [12]. We will assume that the correlation matrices $\mathbf{R}_T$ and $\mathbf{R}_R$ are both *full rank*. This assumption is reasonable because wireless systems experiencing channels where $\mathbf{R}_T$ or $\mathbf{R}_R$ are not full rank can be easily reformulated into a model with an equivalent rank $\min(\mathrm{rank}\,(\mathbf{R}_R), \mathrm{rank}\,(\mathbf{R}_T))$ channel. We also assume that the channel is constant for several codeword periods before changing independently.

The precoding matrix $\mathbf{F}$ is defined by a function f : $\mathbb{C}^{M_r \times M_t} \rightarrow \mathcal{F}$, where $\mathcal{F} = \{\mathbf{F}_1,\ \mathbf{F}_2,\ \ldots,\ \mathbf{F}_N\} \subset \mathbb{C}^{M_t \times M}$,

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[1]We use $|\cdot|$ to denote the absolute value, * to denote matrix conjugate transposition, ^{T} to denote matrix transposition, $\mathbf{I}_k$ to denote the $k \times k$ identity matrix, $\mathbf{e}_i$ to denote the ith column of the matrix $\mathbf{E}$, $\lambda_i\{\mathbf{A}\}$ to denote the ith largest singular value of $\mathbf{A}$, $\mathrm{rank}\,(\cdot)$ to denote a function that computes the rank of a matrix, $\arg\max$ returns a single global maximizer, $\mathcal{U}(m,n)$ (with $m \geq n$) to denote the set of $m \times n$ complex matrices with orthonormal columns, $\|\cdot\|_2$ to denote the matrix two-norm, and $\|\cdot\|_F$ to denote the matrix Frobenius norm.

of the instantaneous channel (i.e., $\mathbf{F} = f(\mathbf{H})$). Assuming perfect pulse-shaping, sampling, and Gaussian noise, the received matrix will be described by

$$\mathbf{Y} = \sqrt{\frac{\rho}{M}}\mathbf{HFC} + \mathbf{V} \tag{2}$$

where $\mathbf{Y}, \mathbf{V} \in \mathbb{C}^{M_r \times T}$ and the entries of the noise matrix $\mathbf{V}$ are assumed to be independent, in space and time, and distributed according to $\mathcal{CN}(0,1)$. Notice that the codeword index in (2) has been suppressed because we are only interested in codeword-by-codeword decoding of $\mathbf{C}(k)$. The number ρ represents the average signal-to-noise ratio (SNR).

OSTBCs offer the benefit of an optimal *single-dimensional* maximum-likelihood (ML) detection (i.e., each of the symbols $s_1, s_2, \ldots, s_{n_s}$ can detected independently). The linear processing required for this detection decomposes the received matrix $\mathbf{Y}$ into M different received symbols. Each received symbol has an average SNR of [13]

$$\text{SNR} = \frac{\rho}{M}\|\mathbf{HF}\|_F^2.$$

The factor $\|\mathbf{HF}\|_F^2$ is the *transmit diversity (TD) channel gain.* This TD channel gain functions in the same manner as that found in traditional beamforming and combining techniques (see, for example, [14] and the references therein). It has been shown (see [2]–[5]) that the probability of error conditioned on the channel of a precoded OSTBC can be bounded by

$$\Pr(\text{ERROR} \mid \mathbf{H}) \leq \exp\left(-\gamma\|\mathbf{HF}\|_F^2\right)$$

where γ is a positive constant that depends on M, ρ, and $\mathcal{S}$ [13]. The number γ is fixed during transmission, so the precoder selection function f that minimizes the bound on the conditional probability of error is defined by

$$f(\mathbf{H}) = \arg\max_{\mathbf{F}' \in \mathcal{F}} \|\mathbf{HF}'\|_F^2. \tag{3}$$

The analysis in this letter will assume that the precoder matrix is selected from $\mathcal{F}$ using (3).

III. Diversity Performance

Given the selection function in (3), we now address the diversity order obtained by precoded OSTBCs. The theorem will use an $M_t \times NM$ matrix $\mathbf{E}$ formed by concatenating all possible precoder matrices.

Theorem 1: The precoded OSTBC obtains a diversity of order $M_t M_r$ if and only if the matrix $\mathbf{E} = [\mathbf{F}_1\ \mathbf{F}_2\ \cdots\ \mathbf{F}_N]$ is full rank.

Proof: We will first prove the "only if" part of the statement using the contrapositive. Suppose that $\mathbf{E}$ *is not* full rank.

Note that

$$\max_{\mathbf{F}' \in \mathcal{F}} \|\mathbf{HF}'\|_F^2 \leq M \max_{\mathbf{F}' \in \mathcal{F}} \|\mathbf{HF}'\|_2^2 \leq M\|\mathbf{HE}\|_2^2. \tag{4}$$

Let $\mathbf{R}_T^{1/2}\mathbf{E} = \mathbf{U}_L\mathbf{\Sigma}\mathbf{U}_R^*$ where $\mathbf{U}_L \in \mathcal{U}(M_t, M_t)$, $\mathbf{\Sigma}$ is an $M_t \times NM$ diagonal matrix with entry $\lambda_i\{\mathbf{R}_T^{1/2}\mathbf{E}\}$ at entry (i,i), and $\mathbf{U}_R \in \mathcal{U}(NM, NM)$. Because of the rank deficiency in $\mathbf{E}$, $\mathbf{\Sigma}$ will be of the form $\mathbf{\Sigma} = [\overline{\mathbf{\Sigma}}^T\ \mathbf{0}]^T$ where $\overline{\mathbf{\Sigma}}$ consists of the first $\text{rank}\,(\mathbf{E})$ rows of $\mathbf{\Sigma}$ and $\mathbf{0}$ is a zero matrix. Substitution of the singular value decomposition of $\mathbf{R}_T^{1/2}\mathbf{E}$ and (1) in to (4) gives

$$\begin{aligned}\max_{\mathbf{F}' \in \mathcal{F}} \|\mathbf{HF}'\|_F^2 &\leq M\|\mathbf{R}_R\|_2\|\mathbf{H}_w\mathbf{U}_L\mathbf{\Sigma}\mathbf{U}_R^*\|_2^2 \\ &= M\|\mathbf{R}_R\|_2\|\mathbf{H}_w\mathbf{U}_L[\overline{\mathbf{\Sigma}}^T\ \mathbf{0}]^T\|_2^2 \\ &= M\|\mathbf{R}_R\|_2\|\overline{\mathbf{H}_w\mathbf{U}_L}\ \overline{\mathbf{\Sigma}}\|_2^2 \\ &\leq M\|\mathbf{R}_R\|_2\|\mathbf{R}_T^{1/2}\mathbf{E}\|_2^2\|\overline{\mathbf{H}_w\mathbf{U}_L}\|_F^2 \end{aligned} \tag{5}$$

where $\overline{\mathbf{H}_w\mathbf{U}_L}$ is the first $\text{rank}\,(\mathbf{E})$ columns of $\mathbf{H}_w\mathbf{U}_L$ and (5) follows from using the submultiplicative property of a matrix norm and of noting that $\|\overline{\mathbf{\Sigma}}\|_2^2 = \|\mathbf{R}_T^{1/2}\mathbf{E}\|_2^2$.

Note that the matrix $\mathbf{H}_w\mathbf{U}_L$ is equivalent in distribution to $\mathbf{H}_w$ [15]. Therefore, we have upper-bounded the effective TD channel gain by the effective TD channel gain of an $rank(\mathbf{E}) \times M_r$ multi-antenna system experiencing matrix Rayleigh fading. A transmit diversity system with the effective channel gain given by this upper-bound would obtain a diversity order of $\text{rank}\,(\mathbf{E}) \cdot M_r < M_t M_r$ [13]. Therefore, we can conclude that if $\mathbf{E}$ is not full rank then the precoded OSTBC *does not* obtain full diversity order.

Now we will prove the converse. Suppose $\mathbf{E}$ is full rank. Then

$$\max_{\mathbf{F}' \in \mathcal{F}} \|\mathbf{HF}'\|_F^2 \geq \max_{1 \leq i \leq NM} \|\mathbf{He}_i\|_2^2 \geq \frac{1}{M_r}\|\mathbf{HE}\|_1^2.$$

Once again let $\mathbf{R}_T^{1/2}\mathbf{E}$ have a singular value decomposition of $\mathbf{R}_T^{1/2}\mathbf{E} = \mathbf{U}_L\mathbf{\Sigma}\mathbf{U}_R^*$. By substitution

$$\begin{aligned}\max_{\mathbf{F}' \in \mathcal{F}} \|\mathbf{HF}'\|_F^2 &\geq \frac{1}{M_r}\|\mathbf{HE}\|_1^2 \\ &= \frac{1}{M_r}\|\mathbf{R}_R^{1/2}\mathbf{H}_w\mathbf{R}_T^{1/2}\mathbf{E}\|_1^2 \\ &\geq \frac{1}{NMM_r}\|\mathbf{R}_R^{1/2}\mathbf{H}_w\mathbf{U}_L\mathbf{\Sigma}\|_1^2 &(6) \\ &\geq \frac{\lambda_{M_t}^2\{\mathbf{R}_T^{1/2}\mathbf{E}\}}{NMM_r}\|\mathbf{R}_R^{1/2}\mathbf{H}_w\mathbf{U}_L\|_1^2 \\ &\geq \frac{\lambda_{M_t}^2\{\mathbf{R}_T^{1/2}\mathbf{E}\}}{NMM_rM_t^2}\|\mathbf{R}_R^{1/2}\mathbf{H}_w\mathbf{U}_L\|_F^2 &(7)\end{aligned}$$

where (6) uses the fact that $\|\mathbf{AU}\|_1^2 \geq (1/n)\|\mathbf{A}\|_1^2$ when $\mathbf{A} \in \mathbb{C}^{m \times n}$ and $\mathbf{U} \in \mathcal{U}(n,n)$ [16].

Therefore, we have lower-bounded the effective TD channel gain by the effective TD channel gain of an $M_t \times M_r$ transmit diversity system. $\mathbf{H}_w\mathbf{U}_L$ is equivalent in distribution to an uncorrelated matrix Rayleigh fading channel [15]. A system with a TD gain given by (7) has $M_t M_r$ diversity order. We can conclude that the precoded OSTBC obtains full diversity order [13]. ■

This proof gives new insight into the diversity order of precoded OSTBCs. In particular, if the requirement of the precoded system is to obtain full $M_t \times M_r$ diversity performance a minimum of $\lceil \log_2(M_t) - \log_2(M) \rceil$ bits of feedback are needed. It was shown in [14] that limited feedback beamforming requires at least $\lceil \log_2(M_t) \rceil$ bits of feedback to achieve full diversity order. Using precoded OSTBCs saves *at least* $\lfloor \log_2(M) \rfloor$ bits of feedback. Thus simply employing a limited feedback precoded Alamouti code instead of a limited feedback beamforming saves one bit of feedback. This illustrates the benefits of precoded OSTBCs with *minimum* feedback. In particular, we will explore the fact that the antenna selection requirements presented in [5] can be weakened while maintaining full diversity order.

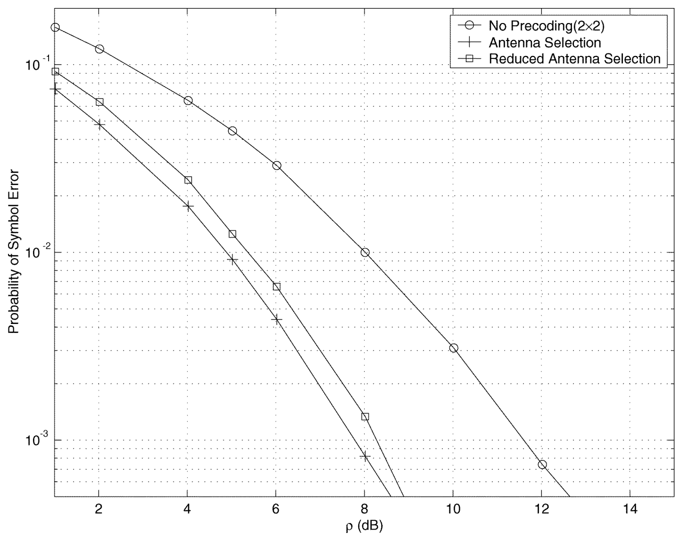

Fig. 1. Probability of symbol error comparison of antenna subset selection schemes for a two substream 4×2 system using 4-QAM.

IV. Reduced Complexity Antenna Subset Selection

In [5], antenna subset selection was considered as a precoding scheme where $\mathcal{F}$ consisted of the $\binom{M_t}{M}$ matrices corresponding to each of the ways of choosing M different columns from $\mathbf{I}_{M_t}$. Antenna subset selection in this form is easily implemented for small values of M_t. For large antenna arrays, however, it is important to find practical methods for obtaining full diversity order using only a limited number of subset combinations.

For this reason, we propose a reduced complexity variant of traditional antenna subset selection. The basic idea is to *use the minimum number of subset configurations to guarantee full diversity order*. This corresponds to $\lceil M_t/M \rceil$ subsets. The supported subset configurations can be designed by arbitrarily choosing $\lceil M_t/M \rceil$ different $M_t \times M$ matrices with columns from $\mathbf{I}_{M_t}$ such that the corresponding $\mathbf{E}$ is full rank. For example, a four transmit antenna system transmitting an Alamouti code on two selected antennas could use the set

$$\mathcal{F} = \left\{ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}^T, \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^T \right\}. \quad (8)$$

Theorem 1 guarantees that reduced complexity antenna subset selection will have full diversity order. There will always be an array gain over an unprecoded $M \times M_r$ OSTBC because

$$\max_{\mathbf{F}' \in \mathcal{F}} \|\mathbf{H}\mathbf{F}'\|_F^2 \geq \|\overline{\mathbf{H}}\|_F^2 \quad (9)$$

where $\overline{\mathbf{H}}$ is the first M columns of $\mathbf{H}$ and would be equivalent to always selecting the first M antennas for transmission.

The complexity savings with our proposed system are dramatic compared with standard antenna subset selection. For example, a system with $M_t = 12$ and $M = 4$ using standard antenna selection requires 165 times as many antenna subset combinations as the reduced complexity subset selection. This large savings comes with the guarantee of full diversity order.

We simulated a 4×2 antenna subset selection system using an Alamouti space-time code transmitting over an uncorrelated, Rayleigh fading channel. The plot in Fig. 1 shows an unprecoded 2×2 Alamouti code, a full complexity antenna subset selection system, and a reduced complexity antenna subset selection system. Note that the reduced complexity technique is within 0.5 dB of the full complexity antenna subset selection. The reduced complexity antenna subset selection can be implemented in a system where the transmitter has no channel knowledge using only one bit of feedback compared with the three bits required for full complexity antenna subset selection.

V. Conclusion

We have presented necessary and sufficient conditions for full diversity order in correlated matrix Rayleigh fading channels for limited feedback precoded OSTBCs. We proved that limited feedback precoded OSTBCs can achieve full diversity order using fewer bits than limited feedback beamforming. The analysis also led to the proposal of reduced complexity antenna subset selection. This form of subset selection dramatically reduces the hardware complexity of traditional antenna selection (see [5], [6]) while still maintaining full diversity order.

References

[1] S. M. Alamouti, "A simple transmit diversity technique for wireless communications," *IEEE J. Select. Areas Commun.*, vol. 16, pp. 1451–1458, Oct. 1998.

[2] V. Tarokh, H. Jafarkhani, and A. R. Calderbank, "Space-time block codes from orthogonal designs," *IEEE Trans. Inform. Theory*, vol. 45, pp. 1456–1467, July 1999.

[3] G. Jöngren, M. Skoglund, and B. Ottersten, "Combining beamforming and orthogonal space-time block coding," *IEEE Trans. Inform. Theory*, vol. 48, pp. 611–627, Mar. 2002.

[4] E. G. Larsson, G. Ganesan, P. Stoica, and W.-H. Wong, "On the performance of orthogonal space-time block coding with quantized feedback," *IEEE Commun. Lett.*, vol. 6, pp. 487–489, Nov. 2002.

[5] D. A. Gore and A. J. Paulraj, "MIMO antenna subset selection with space-time coding," *IEEE Trans. Signal Processing*, vol. 50, pp. 2580–2588, Oct. 2002.

[6] W.-H. Wong and E. G. Larsson, "Orthogonal space-time block coding with antenna selection and power allocation," *Electron. Lett.*, vol. 39, pp. 379–381, Feb. 2003.

[7] G. Jöngren and M. Skoglund, "Utilizing quantized feedback information in orthogonal space-time block coding," in *Proc. IEEE Global Telecommunications Conf.*, vol. 2, Nov.-Dec. 2000, pp. 995–999.

[8] G. Ganesan, P. Stoica, and E. G. Larsson, "Diagonally weighted orthogonal space-time block codes," in *Proc. 36th Asilomar Conf. on Signal, Systems and Computers*, vol. 2, Nov. 2002, pp. 1147–1151.

[9] T. Xiaofeng, H. Harald, Y. Zhuizhuan, Q. Haiyan, and Z. Ping, "Closed loop space-time block code," in *Proc. IEEE Vehicular Technology Conf.*, vol. 2, Oct. 2001, pp. 1093–1096.

[10] D. Chizhik, F. Rashid-Farrokhi, J. Ling, and A. Lozano, "Effect of antenna separation on the capacity of BLAST in correlated channels," *IEEE Commun. Lett.*, vol. 4, pp. 337–339, Nov. 2000.

[11] L. Schumacher, J. P. Kermoal, F. Frederiksen, K. I. Pedersen, A. Algans, and P. E. Mogensen, "MIMO channel characterisation," Tech. Rep., European IST-1999-11729 Project METRA, Feb. 2001.

[12] V. Erceg, L. Schumacher, P. Kyritsi, A. Molisch, D. S. Baum, A. Y. Gorokhov, C. Oestges, V. J. Rhodes, J. Medbo, D. Michelson, M. Webster, E. Jacobsen, D. Cheung, Q. Li, C. Prettie, M. Ho, K. Yu, L. Jengx, A. Jagannatham, N. Tal, and C. Lanzl, "*Indoor MIMO WLAN channel models*," IEEE 802.11-03/161r2, Sept. 2003.

[13] E. G. Larsson and P. Stoica, *Space-Time Block Coding for Wireless Communications*. New York: Cambridge Univ. Press, 2003.

[14] D. J. Love and R. W. Heath, Jr., "Necessary and sufficient conditions for full diversity order in correlated Rayleigh fading beamforming and combining systems," *IEEE Trans. Wireless Commun.*, to be published.

[15] A. T. James, "Distributions of matrix variates and latent roots derived from normal samples," *Ann. Math. Stat.*, vol. 35, pp. 475–501, June 1964.

[16] D. J. Love, R. W. Heath, Jr., and T. Strohmer, "Grassmannian beamforming for multiple-input multiple-output wireless systems," *IEEE Trans. Inform. Theory*, vol. 49, pp. 2735–2747, Oct. 2003.