Grassmannian Beamforming for Multiple-Input Multiple-Output Wireless Systems

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***Abstract*— Multiple-input multiple-output (MIMO) wireless systems provide capacity much larger than that provided by traditional single-input single-output (SISO) wireless systems. Beamforming is a low complexity technique that increases the receive signal-to-noise ratio (SNR), however, it requires channel knowledge. Since in practice channel knowledge at the transmitter is difficult to realize, we propose a technique where the receiver designs the beamforming vector and sends it to the transmitter by transmitting a label in a finite set, or codebook, of beamforming vectors. A codebook design method for quantized versions of maximum ratio transmission, equal gain transmission, and generalized selection diversity with maximum ratio combining at the receiver is presented. The codebook design criterion exploits the quantization problem's relationship with Grassmannian line packing. Systems using the beamforming codebooks are shown to have a diversity order of the product of the number of transmit and the number of receive antennas. Monte Carlo simulations compare the performance of systems using this new codebook method with the performance of systems using previously proposed quantized and unquantized systems.**

I. Introduction

One low complexity technique for obtaining the spatial diversity advantage provided by multiple antenna wireless systems is transmit beamforming and receive combining [1]. Optimal combining has long been used in receive diversity systems for single-input multiple-output (SIMO) wireless systems [2]. When the transmitter has channel knowledge, the extension of SIMO combining methods to multiple-input single-output (MISO) beamforming is straightforward because both systems encounter vector fading channels. The design methods used for vector channels, however, can not be directly applied to the matrix channels encountered in multiple-input multiple-output (MIMO) wireless systems [3],[4] because the entries can not be directly co-phased. Transmission techniques that provide maximum receive SNR solutions given constraints on the beamforming vector such as maximum ratio transmission (MRT) [5], equal gain transmission (EGT) [3], [4], and selection diversity transmission (SDT) [6] have all been extended to the case of MIMO wireless systems.

Beamforming techniques require channel information at the transmitter. SDT and generalized selection diversity transmission (GSDT), a technique where the receiver chooses subsets of transmit antennas based on channel conditions, are the only MIMO beamforming and combining methods that have a straightforward digital implementation. MRT and EGT both require, in theory, infinite precision feedback. Since infinite resolution is impossible, it is preferable to quantize the beamforming vectors that the receiver sends to the transmitter. Limited feedback equal gain beamforming was explored in [3], [4], [7] and called quantized equal gain transmission (QEGT). Quantized maximum ratio transmission (QMRT) systems were addressed in [8]–[10] for the MISO case. Codebooks were designed using vector quantization techniques in [8], [9], but a specific design criteria was not developed. Unlike previously proposed work [8]–[10], we consider explicitly quantized beamforming for MIMO systems. Further, we also consider the case where side constraints are imposed on the beamforming vector.

This paper develops design criteria for quantized maximum ratio transmission, quantized equal gain transmission, and generalized selection diversity transmission codebooks for independent identically distributed (i.i.d.) MIMO Rayleigh fading channels. We show that the quantized beamformer should use a codebook designed using a phase-invariant mean squared error distortion function. A codebook design criterion is developed using a probabilistic analysis of the the optimal maximum ratio transmission vector. This analysis ties the quantization of this vector to Grassmannian line packing.

This paper is organized as follows. Section II overviews MIMO beamforming systems. Grassmannian line packing and spherical codes are discussed in Section III. Section IV looks at the distribution of an optimal beamforming vector and ties the probabilistic results to Grassmannian line packing. The performance of quantized beamforming systems is analyzed in Section V. Section VI presents Monte Carlo simulation results. The paper concludes in Section VII.

II. System Overview

A MIMO system with transmit beamforming and receive combining is shown in Fig. 1 with M_t transmit antennas and M_r receive antennas. A symbol s ($s \in \mathbb{C}$, the field of complex numbers) to be transmitted is multiplied by weight w_l ($w_l \in \mathbb{C}$) at the l^{th} ($1 \leq l \leq M_t$) transmit antenna. The data received by

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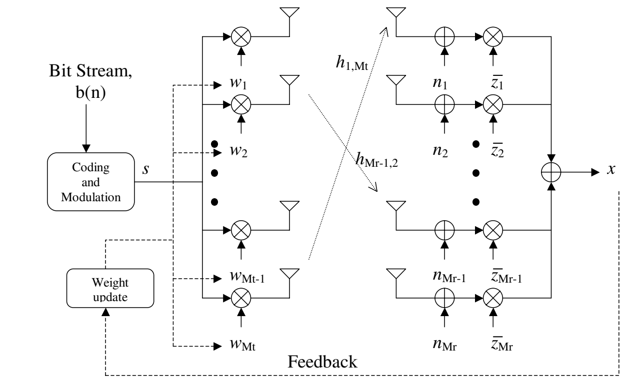

Fig. 1. Block diagram of a MIMO system.

the k^{th} ($1 \leq k \leq M_r$) receive antenna, sent by the l^{th} transmit antenna, is multiplied a channel gain $h_{k,l}$ with $h_{k,l}$ is distributed according to $\mathcal{CN}(0,1)$. This paper assumes $h_{k,l}$ is independent of $h_{m,n}$ if $k \neq m$ or $l \neq n$. This channel model is valid when there is a significant amount of scattering and all antennas are sufficiently spaced. The data received by the k^{th} receive antenna is then added with noise n_k where n_k is distributed according to $\mathcal{CN}(0, N_0)$ and weighted by $\overline{z_k}$ ($z_k \in \mathbb{C}$ with $\bar{\cdot}$ denoting conjugation). The weighted output of each of the M_r receive antennas is summed into the combiner output x.

The input/output relationship can be written in matrix form. Let $\mathbf{w} = [w_1\, w_2 \ldots w_{M_t}]^T$ (with ^{T} denoting matrix transposition), $\mathbf{z} = [z_1\, z_2 \ldots z_{M_r}]^T$, $\mathbf{n} = [n_1\; n_2\; \ldots\; n_{M_r}]^T$, and $\mathbf{H}$ be the $M_r \times M_t$ matrix with coordinate (k,l) equal to $h_{k,l}$. Then the combiner output is $x = (\mathbf{z}^H \mathbf{H}\mathbf{w})s + \mathbf{z}^H\mathbf{n}$ where ^{H} denotes the matrix conjugate transpose.

We would like to design $\mathbf{w}$ and $\mathbf{z}$ to minimize the error rate performance of the system in Fig. 1. It was shown in [3],[4] that minimizing the average probability of symbol error (APSE) requires that we maximize $\mathcal{E}_r$, the receive energy. The formulation above allows us to express $\mathcal{E}_r$ in our system as

$$\begin{aligned} \mathcal{E}_r &= \mathcal{E}_t |\mathbf{z}^H \mathbf{H}\mathbf{w}|^2 / ||\mathbf{w}||_2^2 \\ &= \mathcal{E}_t \mathbf{\Gamma}_r \end{aligned} \tag{1}$$

where $\mathbf{\Gamma}_r$ is the weighted channel gain given by $\mathbf{\Gamma}_r = |\mathbf{z}^H\mathbf{H}\mathbf{w}|^2/||\mathbf{w}||_2^2$, $|\cdot|$ is the absolute value, $||\cdot||_2$ is the induced matrix 2-norm, and $\mathcal{E}_t$ is the transmit energy. Thus to maximize $\mathcal{E}_r$ we must maximize $\mathbf{\Gamma}_r$. To enforce the transmit power constraint, we will require $||\mathbf{w}||_2 = 1$ and assume $\mathcal{E}_t$ held constant. The SNR for this system is given by $\mathcal{E}_t\mathbf{\Gamma}_r/(||\mathbf{z}||_2^2 N_0)$. Therefore without loss of generality we also constrain $||\mathbf{z}||_2 = 1$.

A transmitter where $\mathbf{w}$ maximizes $|\mathbf{z}^H\mathbf{H}\mathbf{w}|$ for an arbitrary $\mathbf{z}$ is called maximum ratio transmission. If $\mathbf{w}$ is restricted to be of the form $\mathbf{w} = \frac{1}{\sqrt{M_t}}[e^{j\theta_1}\; \ldots\; e^{j\theta_{M_t}}]^T$ the beamformer is called equal gain transmission. When $\mathbf{w}$ selects some subset of the M_t antennas and transmits with equal power on each of the selected antennas, this is called generalized selection diversity transmission. An example of a GSDT system for a two transmit antenna case would be to construct $\mathbf{w}$ to be in the set $\{[1\; 0]^T, [0\; 1]^T, \frac{1}{\sqrt{2}}[1\; 1]^T\}$. In MIMO systems, the receive vector also needs to be chosen. A receiver where $\mathbf{z}$ maximizes $|\mathbf{z}^H\mathbf{H}\mathbf{w}|$ given $\mathbf{w}$ is called a maximum ratio combiner (MRC). This paper assumes the receiver uses MRC.

Given no design constraints on the form of $\mathbf{w}$ or $\mathbf{z}$, the optimal solutions in an APSE sense are the beamforming and combining unit vectors that maximize $\mathcal{E}_r$. For a combining scheme that solves for transmit weight vector $\mathbf{w}$ using the solution set $\mathcal{W}$ ($\mathcal{W} \subseteq \Omega_{M_t}$ with Ω_{M_t} denoting the set of unit vectors in $\mathbb{C}^{M_t}$) with an MRC receiver, $\mathbf{w}$ is given by [11]

$$\mathbf{w} \in \arg\max_{\mathbf{x}\in\mathcal{W}} ||\mathbf{H}\mathbf{x}||_2 \tag{2}$$

where $\arg\max$ is defined to return the set of global maximizers. This result follows since an optimal combining vector $\mathbf{z}$ for any beamforming vector $\mathbf{w}$ is $\mathbf{z} = \mathbf{H}\mathbf{w}/||\mathbf{H}\mathbf{w}||_2$.

This paper considers the scenario where the transmitter has no knowledge of the channel, so the receiver must send $\mathbf{w}$ to the transmitter. Since $\mathbf{w}$ can be any unit vector in a possibly large solution set (Ω_{M_t} for MRT) it is necessary to introduce some method of quantization. A solution is to let the receiver and transmitter both use a codebook of N beamforming vectors. This allows the receiver to replace the computationally complex singular value decomposition (SVD) of $\mathbf{H}$ needed for an MRT/MRC system (a system using MRT at the transmitter and MRC at the receiver) and the difficult optimization of EGT/MRC [3], [4] with a simple brute force search of the N vectors. The main benefit of choosing $\mathbf{w}$ from an N vector codebook is that the number of bits sent back can be kept to a reasonable number given by $\lceil \log_2 N \rceil$. Unfortunately, it is not obvious how to design an N vector codebook.

III. Grassmannian Line Packing & Spherical Codes

Grassmannian line packing represents one of the famous unsolved applied mathematics problems. Grassmannian line packing techniques can be used to construct vector codebooks called spherical codes. In this section, we therefore give a brief overview of Grassmannian line packing and spherical codes. These topics will be of use in understanding the analysis of beamforming codebooks done in later sections.

The Grassmannian space $\mathcal{G}(m,n)$ is the set of all n-dimensional subspaces of the space $\mathcal{M}^m$, where $\mathcal{M} = \mathbb{R}$ or $\mathcal{M} = \mathbb{C}$. Since MIMO matrix channels have complex entries, we will only consider $\mathcal{M} = \mathbb{C}$. We are also interested in using Grassmannian subspace packing for designing beamforming vector codebooks, therefore we will deal exclusively with $\mathcal{G}(m,1)$.

The *Grassmannian subspace packing problem* is the problem of finding the best packing of N n-dimensional subspaces in $\mathcal{M}^m$. In other words, we want to find N points in $\mathcal{G}(m,n)$ so that the minimal distance between any two of the subspaces is as large as possible. Because we are restricting our study to $\mathcal{G}(m,1)$ with $\mathcal{M}=\mathbb{C}$, the subspaces are lines through the origin in $\mathbb{C}^m$ and the packing problem is to arrange the lines such that the angle between any of two of the lines becomes as large as possible [12]. Note that if $\mathbf{w}$ lies on a line then so does $e^{j\phi}\mathbf{w}$ for all $\phi \in [0, 2\pi)$. Thus the problem simplifies down to arranging N vectors so that the magnitude correlation between any two lines is as small as possible.

Let ρ be a function that maps the set of $m \times N$ matrices with complex unit vector columns, $\mathcal{U}_m^N$, to the set of nonnegative reals, given by

$$\rho(\mathbf{W}) = \max_{1 \le k < l \le N} |\mathbf{w}_k^H \mathbf{w}_l| \tag{3}$$

where $\mathbf{w}_k$ is the k^{th} column of $\mathbf{W}$. A set of N lines is called equiangular if $\left|\mathbf{w}_k^H \mathbf{w}_l\right| = \cos(\theta)$ for all k,l with $k \neq l$ with $\theta \in [0, \pi/2]$.

A set of N vectors representing N lines is a type of spherical code. A *spherical code* of dimension m is defined to be a set of points on Ω_m [13]. Every spherical code has a corresponding matrix in $\mathcal{U}_m^N$. We will define a *complex antipodal spherical code* to be a code with the property that if $\mathbf{w}$ is a member of the spherical code then $e^{j\phi}\mathbf{w}$ is a member of the spherical code for all $\phi \in [0, 2\pi)$. A complex antipodal spherical code can be uniquely represented by a set of N non-collinear vectors. A useful measure of a spherical code's quality is its *density.* Let $\mathbf{W}$ be an $m \times N$ matrix whose columns form an N vector complex antipodal spherical code (i.e. $\mathbf{W} \in \mathcal{U}_m^N$). For a $\gamma \in [0,1)$ and $\mathbf{w} \in \Omega_m$, we will define the spherical cap of radius γ around $\mathbf{w}$ as the set

$$\mathcal{C}_{\mathbf{w}}(\gamma) := \left\{\mathbf{x} \in \Omega_m \;\middle|\; |\mathbf{x}^H \mathbf{w}| \ge \gamma\right\}. \tag{4}$$

The maximum angular radius of a complex antipodal spherical code is given by $\theta_r := \arccos(\rho(\mathbf{W}))/2$.

The density of an N vector spherical code as

$$\begin{aligned}\Delta(\mathbf{W}) &:= \frac{\sum_{k=1}^N S\left(\mathcal{C}_{\mathbf{w}_k}\left(\cos\left(\theta_r\right)\right)\right)}{S\left(\Omega_m\right)} \\ &= \frac{NS\left(\mathcal{C}_{\mathbf{w}_1}\left(\cos\left(\theta_r\right)\right)\right)}{S\left(\Omega_m\right)}\end{aligned} \tag{5}$$

where S is a function that gives the surface area of a set of points on the unit sphere and $\Delta : \mathcal{U}_m^N \to [0,1]$. The following theorem states an important property of the density function.

Theorem 1: Let $\mathbf{W}, \widetilde{\mathbf{W}} \in \mathcal{U}_m^N$. $\Delta(\mathbf{W}) > \Delta(\widetilde{\mathbf{W}})$ if and only if $\rho(\mathbf{W}) < \rho(\widetilde{\mathbf{W}})$ [11].

Therefore minimizing the maximum absolute correlation maximizes the density. This important observation will be used in developing a beamforming codebook criteria.

IV. Codebook Analysis and Design

In [14] it is shown that an optimal beamforming vector for MRT systems is the dominant right singular vector of matrix channel $\mathbf{H}$. Therefore, $\mathbf{w}_{MRT}$ that satisfies (2) ($\mathcal{W} = \Omega_{M_t}$) is an optimal MRT solution. The optimal beamforming vector is given by

$$\mathbf{w}_{MRT} \in \arg\max_{\mathbf{x} \in \Omega_{M_t}} |\mathbf{x}^H \mathbf{H}^H \mathbf{H} \mathbf{x}|. \tag{6}$$

Note that if $\mathbf{w}_{MRT}$ satisfies (6), then $e^{j\phi}\mathbf{w}_{MRT}$ also satisfies (6) because $e^{j\phi}$ has unit magnitude. Therefore the beamforming vector is not unique.

We can restate this non-uniqueness property in terms of points on a complex line instead of vectors. Let $\mathcal{P}_{\mathbf{w}}$ denote the column space of the vector $\mathbf{w}$. The subspace $\mathcal{P}_{\mathbf{w}}$ can easily be understood as a line in $\mathbb{C}^{M_t}$ passing through the origin. We will define a *representative vector* of $\mathcal{P}_{\mathbf{w}}$ as any vector in Ω_{M_t} whose column space is equal to $\mathcal{P}_{\mathbf{w}}$. *For any line $\mathcal{P}$ in $\mathbb{C}^{M_t}$ that passes through the origin, all vectors $\mathbf{w} \in (\mathcal{P} \bigcap \Omega_{M_t})$ provide the same receive SNR and the same probability of symbol error.*

In order to understand the properties of the codebook the distribution of the optimal weight vector must now be addressed. The distribution of $\mathbf{X} = \mathbf{H}^H\mathbf{H}$ is the complex Wishart distribution [15] when $\mathbf{H}$ is defined as in Section II. An important property of of complex Wishart distributed random matrices is summarized in Theorem 2 [15].

Theorem 2: If $\mathbf{X}$ is complex Wishart distributed, then $\mathbf{X}$ is equivalent in distribution to $\mathbf{U}\mathbf{D}\mathbf{U}^H$ where $\mathbf{U}$ is uniformly distributed on the group of $M_t \times M_t$ unitary matrices and $\mathbf{D} = diag_{M_t}\{\lambda_1, \ldots, \lambda_{M_x}\}$ ($\lambda_i \ge \lambda_k\ \forall i \ge k$), the diagonal matrix with $\lambda_1, \ldots, \lambda_{M_x}$ on the first M_x diagonal entries and zeros on the remaining diagonal terms with $M_x = \min(M_r, M_t)$, has a distribution found in [15].

One solution to (6) is equivalent in distribution to $\mathbf{w}_{MRT} = \mathbf{U}[1\ 0\ 0\ \cdots\ 0]^T$ with $\mathbf{U}$ given in Theorem 2. The following theorem proven in [11] will allow the distribution of this optimal vector to be understood.

Theorem 3: Let $\mathbf{U}$ be a uniformly distributed $\mathrm{M}_t \times M_t$ random unitary matrix. If $\mathbf{v} \in \Omega_{M_t}$ then $\mathbf{U}\mathbf{v}$ is uniformly distributed on Ω_{M_t}.

Therefore since $\mathbf{U}$ is uniformly distributed on $\mathcal{U}_{M_t}$ and $[1\ 0\ 0\ \cdots\ 0]^T$ is a unit vector, Theorem 3 states that $\mathbf{w}_{MRT} = \mathbf{U}[1\ 0\ 0\ \cdots\ 0]^T$ is distributed uniformly on Ω_{M_t}. The theorem also shows that each of the columns of $\mathbf{U}$ and any unit norm linear combination of columns of $\mathbf{U}$ are each uniformly distributed on Ω_{M_t}.

A codebook will be represented by an $M_t \times N$ matrix $\mathbf{W}$ with unit code vector columns. The i^{th} column of $\mathbf{W}$ will be denoted by $\mathbf{w}_i$. The receiver will choose a beamforming vector from the columns of $\mathbf{W}$ using the encoding function $\mathcal{Q}_{\mathbf{w}} : \mathbb{C}^{M_r \times \mathrm{M}_t} \to \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_N\}$ given by [3]

$$\mathcal{Q}_{\mathbf{w}}(\mathbf{H}) = \arg\max_{\mathbf{x} \in \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_N\}} \|\mathbf{H}\mathbf{x}\|_2^2 \tag{7}$$

in order to maximize the receive SNR. Note that $\mathcal{Q}$ is not solely a function of the maximum singular value direction in the matrix channel case. This follows because certain matrix channels arise where it is better to use the quantized beamforming vector that is close to some linear combination of singular vectors. Channels where several of the singular values are approximately equal would fall into this case.

The expected channel gain is given by the performance metric $G(\mathbf{W}) = E_{\mathbf{H}}\|\mathbf{H}\mathcal{Q}_{\mathbf{w}}(\mathbf{H})\|_2^2$ with $E_{\mathbf{H}}$ denoting expectation over $\mathbf{H}$. This can be rewritten as [11]

$$G(\mathbf{W}) = E_{\mathbf{H}}\left[\max_{0<i\leq N}\sum_{k=1}^{M_t}\lambda_k|\mathbf{v}_k^H\mathbf{w}_i|^2\right] \geq E_{\mathbf{H}}\left[\lambda_1\right]E_{\mathbf{H}}\left[\max_{0<i\leq N}|\mathbf{v}_1^H\mathbf{w}_i|^2\right] \quad (8)$$

where $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{M_t}$ are the M_t eigenvalues and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{M_t}$ are the corresponding eigenvectors of the matrix $\mathbf{H}^H\mathbf{H}$. The set of vectors that maximizes the lower bound in (8) is the set of vectors that minimizes the phase-invariant minimum mean-squared error (MMSE) distortion function for a uniformly distributed unit vector source.

Unfortunately, this vector quantization problem does not have a known optimal solution [3]. In order to address this problem we can lower bound the result in (8) by

$$G(\mathbf{W}) \geq E_{\mathbf{H}}\left[\lambda_1\right]\cos^2\left(\theta_r\right)\Delta(\mathbf{W}). \quad (9)$$

This result follows from the fact that $\mathbf{v}_1$ is distributed uniformly on Ω_{M_t} according to Theorem 3.

Therefore we can attempt to maximize G by maximizing the lower bound and thus the spherical code density. Using Theorem 1 we must therefore minimize the maximum absolute inner product between any pair of codebook vectors. Thus we propose the following criterion for designing Grassmannian beamforming codebooks.

Grassmannian Beamforming Criterion: Design the set of codebook vectors $\{\mathbf{w}_i\}_{i=1}^N$ such that the corresponding codebook matrix $\mathbf{W}$ minimizes $\rho(\mathbf{W}) = \max_{1\leq k<l\leq N}|\mathbf{w}_i^H\mathbf{w}_j|$.

As discussed in Section III, this criterion is exactly the same as spacing representative vectors so that the closest pair of corresponding complex lines has maximum angular separation. This problem is mathematically challenging from an analytical point of view [13] but numerical simulations can also yield good codebooks.

Previous work [16] explored using the *Grassmannian Beamforming Criterion* to design QMRT codebooks. This criterion however can easily extend to constrained beamforming schemes such as equal gain or generalized selection. If we wish to constrain our optimization to the set $\mathcal{V}$ where $\mathcal{V} \subseteq \mathcal{U}_{M_t}^N$, the optimal codebook matrix $\mathbf{W}$ would be given by

$$\mathbf{W} \in \arg\min_{\mathbf{X}\in\mathcal{V}}\rho(\mathbf{X}). \quad (10)$$

QMRT sets $\mathcal{V} = \mathcal{U}_{M_t}^N$, and thus codebook design becomes, for a given M_t and N, finding the matrix in $\mathcal{U}_{M_t}^N$ with the smallest maximum absolute correlation between any two columns. Codebooks of this type can be design analytically when applicable [12], but in most cases computer searches must be employed. Notice that when $N \leq M_t$ maximally spaced packings are easy to find: simply take N columns of any $M_t \times M_t$ unitary matrix.

Examples of cases where $\mathcal{V}$ is a strict subset of $\mathcal{U}_{M_t}^N$ include QEGT and GSDT. Equal gain transmission chooses $\mathbf{W}$ by minimizing $\rho(\mathbf{W})$ over the set $\mathcal{F} = \{\mathbf{W} \in \mathcal{U}_{M_t}^N \mid w_{i,i} = e^{j\theta}/\sqrt{M_t}\}$. A generalized selection diversity transmission system chooses a codebook over the set $\mathcal{I}$, the set of matrices in $\mathcal{U}_{M_t}^N$ such that each column can be written as a normalized sum of columns from the $M_\times M_t$ identity matrix (i.e. transmitting on a subset of antennas). The set $\mathcal{F}$, just as $\mathcal{U}_{M_t}^N$, is uncountable and thus requires some form of iterative optimization. The set $\mathcal{I}$, however, is finite and thus a brute force search can yield a globally optimal $\mathbf{W}$.

V. Codebook Performance

Closed form results on the average probability of symbol error for quantized beamforming systems are difficult if not impossible to determine. We will therefore bound the diversity, the asymptotic slope of the average probability of symbol error curve, for quantized beamforming systems using maximum ratio combining.

Lemma 1: Quantized beamforming and maximum ratio combining systems provide an M_rM_t diversity order as long as the vectors in the beamformer codebook span $\mathbb{C}^{M_t}$.
The proof of Lemma 1 is given in [11], [16].

This guarantee of diversity order is extremely important because MIMO transmissions that do not obtain full diversity are not fully utilizing the spatial diversity provided by sufficiently spaced antennas. Thus the key indicator that must be considered in choosing a codebook size N and beamforming technique is array gain. The maximization of the lower bound in (8) and the *Grassmannian Beamforming Criterion* thus give a technique for increasing the array gain in our quantized beamforming system.

VI. Simulations

All simulations used binary phase shift keying (BPSK) modulation and i.i.d. Rayleigh fading (where $h_{k,l}$ is distributed according to $\mathcal{CN}(0,1)$). The APSE is estimated using at least 1.5 million iterations per SNR point. Codebooks were designed using random search criterion for QEGT and QMRT. GSDT codebooks are globally optimal since searching over all possible codebooks is feasible. Other simulation results are available in [3].

Experiment # 1 In the first experiment, the two different methods of QEGT codebook design are compared on a 3 transmit and 3 receive antenna system. The results are shown in Fig. 2. The old method refers to the codebook design method outlined in [3], [4] with $R = 400$ (R denoting a codebook design input). The new method uses low correlation vector codebooks. A 2 bit QEGT/MRC system using the new code design method provides a 0.25dB coding gain compared to

SDT/MRC; SDT/MRC is proposed in [6]. A 3 bit new design method QEGT codebook outperforms a 5 bit old design method QEGT codebook by around 0.05dB. Performance improves by 0.03dB when changing from 3 bit new QEGT to 5 bit new QEGT.

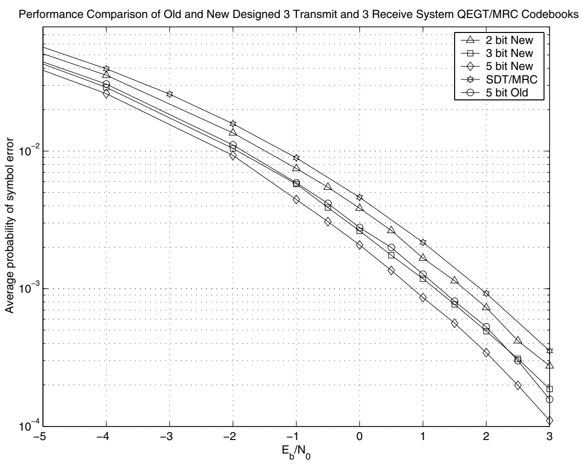

Fig. 2. Comparison of symbol probability of error for 3 transmit and 3 receive antenna systems using QEGT/MRC with the old and new codebook designs.

Experiment # 2 The final experiment, shown in Fig. 3, compares all three of the quantized vector codebooks discussed in this paper for a 4 transmit and 2 receive antenna system. A 4 bit GSDT/MRC system performs around 0.7dB better than SDT/MRC and around 2dB worse than MRT/MRC. A 4 bit QEGT/MRC and 4 bit QMRT/MRC system both perform approximately the same, outperforming a 4 bit GSDT/MRC system by approximately 0.47dB. MRT/MRC outperforms 6 bit QMRT/MRC by approximately 0.9dB and 6 bit QEGT/MRC by approximately 1.2dB.

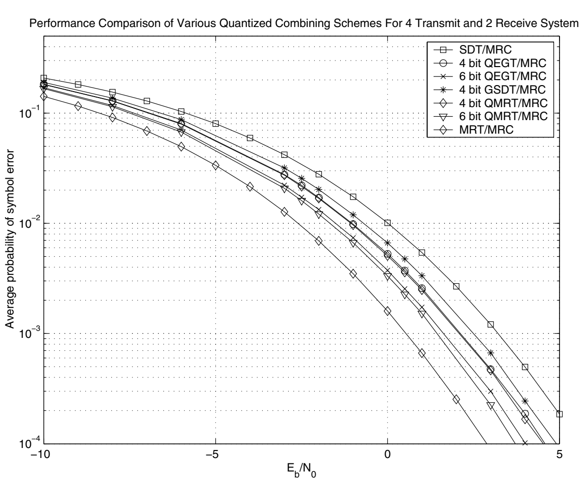

Fig. 3. Comparison of symbol probability of error for 4 transmit and 2 receive antenna systems using QMRT/MRC, GSDT/MRC, and QEGT/MRC.

VII. Conclusion

This paper examined limited feedback beamforming techniques for multi-antenna wireless systems using maximum ratio transmission, equal gain transmission, or generalized selection diversity transmission. The distribution of an optimal beamforming vector for a system using MRT/MRC was found to be uniform on the complex unit sphere. This analysis leads to codebook design criterion using Grassmannian line packing.

There are many problems with multi-antenna feedback systems that have yet to be answered. Probabilistic performance analysis is needed to determine how close the performance of a quantized beamforming can come to optimal performance. It may also be possible to extend the quantization techniques for beamforming system to quantized precoding systems.

References

[1] C. B. Dietrich, Jr., W. L. Stutzman, B.-K. Kim, and K. Dietze, "Smart antennas in wireless communications: base-station diversity and handset beamforming," *IEEE Ant. and Prop. Mag.*, vol. 42, pp. 142–151, Oct. 2000.

[2] W. C. Jakes, *Microwave Mobile Communications*. New York: John Wiley & Sons, 1974.

[3] D. J. Love and R. W. Heath, Jr., "Equal gain transmission in multiple-input multiple-output wireless systems." to appear in *IEEE Trans. Comm.*

[4] D. J. Love and R. W. Heath, Jr., "Equal gain transmission in multiple-input multiple-output wireless systems," in *Proc. of IEEE Globecom*, 2002.

[5] T. K. Y. Lo, "Maximum ratio transmission," *IEEE Trans. Comm.*, vol. 47, pp. 1458–1461, Oct. 1999.

[6] S. Thoen, L. Van der Perre, B. Gyselinckx, and M. Engels, "Performance analysis of combined transmit-SC/receive-MRC," *IEEE Trans. Comm.*, vol. 49, pp. 5–8, January 2001.

[7] R. W. Heath, Jr. and A. Paulraj, "A simple scheme for transmit diversity using partial channel feedback," in *Proc. of IEEE Annual Asilomar Conf. on Signals, Systems, and Computers*, November 1998.

[8] A. Narula, M. J. Lopez, M. D. Trott, and G. W. Wornell, "Efficient use of side information in mulitiple-antenna data transmission over fading channels," *IEEE Jour. Select. Areas in Comm.*, vol. 16, pp. 1423–1436, October 1998.

[9] K. K. Mukkavilli, A. Sabharwal, and B. Aazhang, "Design of multiple antenna coding schemes with channel feedback," in *Proc. of IEEE Asilomar Conf. on Signals, Systems, and Computers*, pp. 1009–1013, Nov. 2001.

[10] K. K. Mukkavilli, A. Sabharwal, E. Erkip, and B. Aazhang, "On beamforming with finite rate feedback in multiple antenna systems." submitted to *IEEE Trans. Inf. Th.* in Oct. 2002.

[11] D. J. Love, R. W. Heath, Jr., and T. Strohmer, "Grassmannian beamforming for multiple-input multiple-output wireless systems." Submitted to *IEEE Trans. Inf. Th.* in Oct. 2002.

[12] T. Strohmer and R. W. Heath, Jr., "Grassmannian frames with applications to coding and communications." Submitted to *Applied and Computational Harmonic Analysis* in July 2002.

[13] J. H. Conway and N. J. A. Sloane, *Sphere Packings, Lattices, and Groups*. New York: Springer-Verlag, 1993.

[14] P. A. Dighe, R. K. Mallik, and S. S. Jamuar, "Analysis of a transmit-receive diversity in Rayleigh fading," in *Proc. of IEEE Globecom*, vol. 2, pp. 1132–1136, November 2001.

[15] A. Edelman, *Eigenvalues and condition numbers of random matrices*. PhD thesis, Massachusetts Institute of Technology, 1989.

[16] D. J. Love, R. W. Heath, Jr., and T. Strohmer, "Quantized maximum ratio transmission for multiple-input multiple-output wireless systems," in *Proc. of IEEE Asilomar Conf. on Signals, Systems, and Computers*, Nov. 2002.