Lecture 31

- Phasors: KVL, KCL, & Ohm's phasor law

§10.5 – 10.6

Phasors

Today, we are going to introduce "Phasor" representation of voltage and current for efficient sinusoidal steady state analysis.

Consider a sinusoidal voltage $V(t)$, it can be written as

magnitude ; initial phase

$$\boxed{V(t) = V_0 \cos(\omega t + \theta)}$$

From Euler's identity: $e^{j\alpha} = \cos\alpha + j\sin\alpha$

thus, $\cos\alpha = Re[e^{j\alpha}]$

We have

$$V(t) = Re\left[V_0 e^{j(\omega t + \theta)}\right]$$

$$= Re\left[V_0 e^{j\theta} e^{j\omega t}\right] \qquad (1)$$

$V(t)$'s phasor representation.

it is complex valued, and it does not depend on time.

Denote $v(t)$'s phasor representation by $\tilde{V}$, then it is defined as

$$\boxed{\text{phasor: } \quad \tilde{V} = V_0 e^{j\theta}}$$

a complex quantity having V_0 as magnitude and θ as phase.

From ①, we also see the relationship between $v(t)$ and its phasor

$$\boxed{v(t) = Re[\tilde{V} e^{j\omega t}]}$$

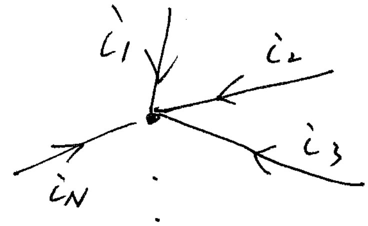
So if we can find $\tilde{V}$, we multiply it by $e^{j\omega t}$, then take real part, we can get its time-domain real-valued form $v(t)$.

Similarly, any sinusoidal current $\underset{I\cos(\omega t+\theta)}{\overset{\|}{i(t)}}$ can be written as

$$\boxed{i(t) = Re[\tilde{I} e^{j\omega t}]}$$

where phasor $\tilde{I} = I e^{j\theta}$.

KCL for phasors



KCL: $\boxed{i_1 + i_2 + i_3 + \cdots + i_N = 0}$

For sinusoidal currents,

$$I_1 \cos(\omega t + \theta_1) + I_2 \cos(\omega t + \theta_2) + \cdots + I_N \cos(\omega t + \theta_N) = 0$$

$$Re[\underbrace{I_1 e^{j\theta_1}}_{i_1(t)\text{'s phasor}} e^{j\omega t}] + Re[\underbrace{I_2 e^{j\theta_2}}_{i_2(t)\text{'s phasor}} e^{j\omega t}] + \cdots + Re[\underbrace{I_N e^{j\theta_N}}_{i_N(t)\text{'s phasor}} e^{j\omega t}] = 0$$

$$Re(\tilde{I}_1 e^{j\omega t}) + Re(\tilde{I}_2 e^{j\omega t}) + \cdots + Re[\tilde{I}_N e^{j\omega t}] = 0$$

$$Re[(\tilde{I}_1 + \tilde{I}_2 + \cdots + \tilde{I}_N) e^{j\omega t}] = 0$$

the above is true for any time t,

$$\therefore \quad \boxed{\tilde{I}_1 + \tilde{I}_2 + \cdots + \tilde{I}_N = 0} \quad (*)$$

Proof: choose t such that $\omega t = 0$, then

$$Re[(\tilde{I}_1 + \tilde{I}_2 + \cdots + \tilde{I}_N)] = 0 \quad (a)$$

choose another t such that $\omega t = \frac{\pi}{2}$, then

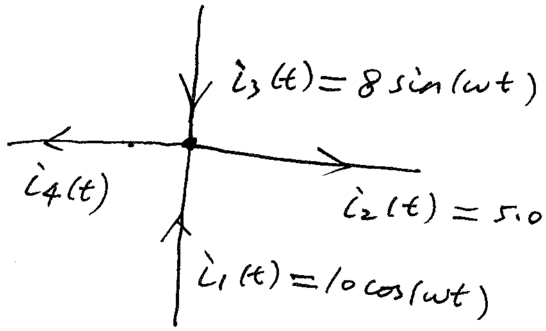
$$Re[(\tilde{I}_1 + \tilde{I}_2 + \cdots + \tilde{I}_N) j] = 0 \quad (b)$$

(b) means the $Im[\tilde{I}_1 + \cdots + \tilde{I}_N] = 0$

So (a) and (b) yields $(*)$

KCL holds true for phasors!

Example 1 Find $i_4(t)$ and write it as a single sinusoidal function.



Solution We can use traditional method

$i_4 = i_3 - i_2 + i_1$ to find the answer.

We can also use phasor method.

- $i_1(t) = 10\cos(\omega t)$

  Its phasor $\tilde{I}_1 = 10e^{j0°} = 10$

- $i_2(t) = 5.043\cos(\omega t + 7.52°)$

  Its phasor $\tilde{I}_2 = 5.043e^{j7.52°} = 5.043(\cos 7.52° + j\sin 7.52°)$

  $\doteq 5 + j0.66$

- $i_3(t) = 8\sin(\omega t) = 8\cos(\omega t - 90°)$

  $\tilde{I}_3 = 8e^{-j90°} = -j8$.

- Use phasor KCL, we have

$$\tilde{I}_4 = \tilde{I}_3 - \tilde{I}_2 + \tilde{I}_1 = -j8 - 5 - j0.66 + 10 = \boxed{5 - j8.66}$$

~~$\angle j8$~~

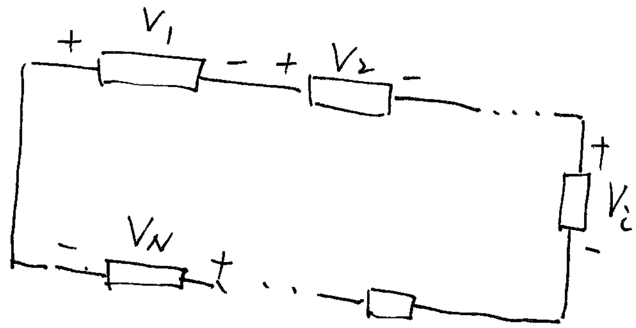
$$= 10\,e^{-j60^\circ}$$

- back to real-valued $i_4(t)$,

$$i_4(t) = Re[\tilde{I}_4 e^{j\omega t}] = Re[10e^{-j60^\circ} e^{j\omega t}]$$

$$= \boxed{10\cos(\omega t - 60^\circ)}$$

KVL for phasors



KVL: $V_1(t) + V_2(t) + \cdots + V_N(t) = 0$

For sinusoidal voltages,

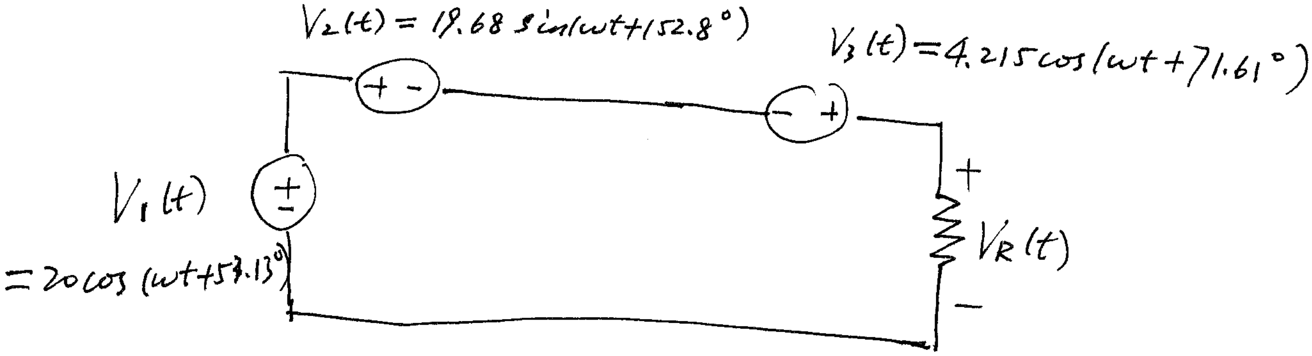
$$\mathrm{Re}[\tilde{V}_1 e^{j\omega t} + \tilde{V}_2 e^{j\omega t} + \cdots + \tilde{V}_N e^{j\omega t}] = 0$$

$$\Rightarrow \mathrm{Re}[(\tilde{V}_1 + \tilde{V}_2 + \cdots + \tilde{V}_N) e^{j\omega t}] = 0$$

$$\Rightarrow \boxed{\tilde{V}_1 + \tilde{V}_2 + \cdots + \tilde{V}_N = 0}$$

KVL also holds true for phasors.

Example 2 Find $V_R(t)$ and write it as a single sinusoidal function



Solution Use phasor representation,

$$\tilde{V}_1 = 20e^{j53.13°} \doteq 12 + j16$$

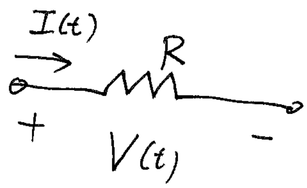
$$\tilde{V}_2 = 19.68e^{j(152.8° - 90°)} \doteq 9 + j17.5$$

$$\tilde{V}_3 = 4.215e^{j71.61°} \doteq 1.33 + j4$$

$$\tilde{V}_R = \tilde{V}_3 - \tilde{V}_2 + \tilde{V}_1 = 4.33 + j2.5 \doteq 5e^{j30°}$$

$$V_R(t) = Re\left[\tilde{V}_R e^{j\omega t}\right] = \boxed{5\cos(\omega t + 30°)}$$

Phasor $\tilde{V}-\tilde{I}$ relationship for resistance R



$$V(t) = R\, I(t)$$

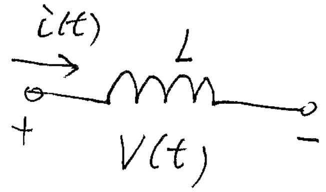
For sinusoidal voltage and current,

$$\mathrm{Re}[\tilde{V} e^{j\omega t}] = R\, \mathrm{Re}[\tilde{I} e^{j\omega t}] = \mathrm{Re}[\tilde{I} R e^{j\omega t}]$$

True for any time t

$$\Longrightarrow \boxed{\tilde{V} = \tilde{I} R}$$

Phasor $\tilde{V} - \tilde{I}$ relationship for inductance L



$$V(t) = L \frac{di(t)}{dt}$$

For sinusoidal voltage and current

$$Re[\tilde{V} e^{j\omega t}] = L \frac{d}{dt} Re[\tilde{I} e^{j\omega t}]$$

$$= L\, Re\left[\frac{d}{dt}(\tilde{I} e^{j\omega t})\right]$$

$$= L\, Re[\tilde{I}(j\omega e^{j\omega t})]$$

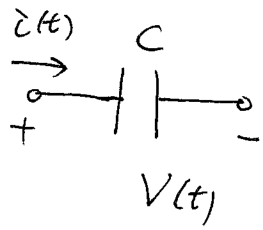
$$= Re[\tilde{I} j\omega L (e^{j\omega t})]$$

$$\Rightarrow \boxed{\tilde{V} = j\omega L \tilde{I}}$$

Important advantage:

Using phasors, the $\tilde{V} - \tilde{I}$ relationship for the inductor becomes a simple "algebraic" relation instead of "differential" relation! This will help us avoid "differential equations" when analyzing circuits involving L.

Phasor $\tilde{V}-\tilde{I}$ relationship for capacitance C



$$i(t) = C\frac{dV(t)}{dt}$$

$$Re[\tilde{I}e^{j\omega t}] = C\frac{d}{dt}Re[\tilde{V}e^{j\omega t}]$$

$$= C\,Re[\frac{d}{dt}(\tilde{V}e^{j\omega t})]$$

$$= C\,Re[j\omega\tilde{V}e^{j\omega t}]$$

$$= Re[\underline{j\omega C\tilde{V}}e^{j\omega t}]$$

$$\Longrightarrow \tilde{I} = j\omega C\tilde{V}$$

$$\Longrightarrow \boxed{\tilde{V} = \frac{\tilde{I}}{j\omega C}}$$

$\boxed{\text{Summary}}$ phasor ohm's Law:

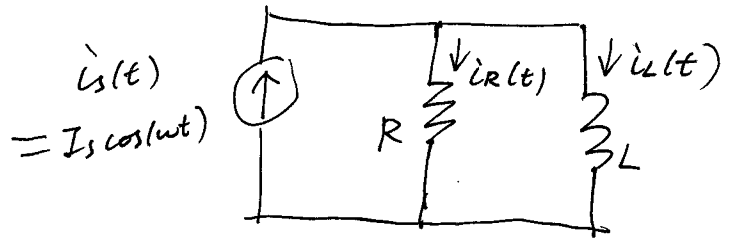
$$\boxed{\tilde{V} = Z\tilde{I}}$$

where impedance $Z = \begin{cases} R & \text{resistance} \\ j\omega L & \text{inductance} \\ \dfrac{1}{j\omega C} & \text{capacitance} \end{cases}$

The reciprocal of Z is called Y (admittance)

$$\therefore \quad Y = \frac{1}{Z} = \begin{cases} \dfrac{1}{R} & \text{resistance} \\ \dfrac{1}{j\omega L} & \text{inductance} \\ j\omega C & \text{capacitance} \end{cases}$$

Example



Compute the SSS response $i_L(t)$.

Solution We solved this problem in last lecture using time-domain differential equation. Let's see whether we can use "phasor method" to simplify the analysis.

We can first find $i_L(t)$'s phasor, denoted by $\tilde{I}_L$, then $i_L(t) = Re[\tilde{I}_L e^{j\omega t}]$

$$\tilde{I}_s = I_s e^{j0^\circ} = I_s$$

$$\tilde{I}_s = \tilde{I}_R + \tilde{I}_L \qquad \text{(KCL for phasors)}$$

$$= \frac{\tilde{V}_L}{R} + \tilde{I}_L \qquad (\tilde{V}_R = \tilde{I}_R R, \text{ and } \tilde{V}_R = \tilde{V}_L)$$

$$= \frac{j\omega L}{R}\tilde{I}_L + \tilde{I}_L$$

$$\Rightarrow \tilde{I}_L = \frac{R}{R+j\omega L}\tilde{I}_s = \frac{R}{R+j\omega L} I_s$$

$$\therefore \tilde{I}_L = \frac{R I_s}{\sqrt{R^2+\omega^2L^2}} e^{j(-\tan^{-1}\frac{\omega L}{R})}$$

$$\therefore i_L(t) = Re[\tilde{I}_L e^{j\omega t}] = \frac{R I_s}{\sqrt{R^2+\omega^2 L^2}}\cos(\omega t - \tan^{-1}\frac{\omega L}{R})$$

the same as we obtained before!