$$\ddot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_r} = \frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_r}\left(\frac{1}{2}\dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i\right)\right) - \frac{\partial}{\partial q_r}\left(\frac{1}{2}\dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i\right)$$

This equation can be proven by obtaining the first term on the right hand side and then the second term (according to Richardson, 1996).

Part 1

The chain rule can be applied first to express the left hand side in terms of expressions containing only the velocity of the position vector:

$$\frac{d}{dt}\left(\dot{\mathbf{r}}_i \bullet \frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_r}\right) = \ddot{\mathbf{r}}_i \cdot \frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_r} + \dot{\mathbf{r}}_i \bullet \frac{d}{dt}\left(\frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_r}\right)$$

$$\ddot{\mathbf{r}}_i \cdot \frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_r} = \frac{d}{dt}\left(\dot{\mathbf{r}}_i \bullet \frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_r}\right) - \dot{\mathbf{r}}_i \bullet \frac{d}{dt}\left(\frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_r}\right)$$

Then the expression in the parenthesis of the first term on the right hand side can be written in terms of the vector product of the velocity vectors:

$$\frac{\partial}{\partial \dot{q}_r}\left(\frac{1}{2}\dot{\mathbf{r}}_i \bullet \dot{\mathbf{r}}_i\right) = \dot{\mathbf{r}}_i \bullet \frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_r}$$

$$\ddot{\mathbf{r}}_i \cdot \frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_r} = \frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_r}\left(\frac{1}{2}\dot{\mathbf{r}}_i \bullet \dot{\mathbf{r}}_i\right)\right) - \dot{\mathbf{r}}_i \bullet \frac{d}{dt}\left(\frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_r}\right)$$

Part 2

The second term in the equation at the top can be proven by expressing the second term in the previous equation in terms of a displacement derivative instead of a velocity derivative:

$$\dot{\mathbf{r}}_i \bullet \frac{d}{dt}\left(\frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_r}\right) = \dot{\mathbf{r}}_i \bullet \frac{d}{dt}\left(\frac{\partial \mathbf{r}_i}{\partial q_r}\right)$$

Then note that $\mathbf{r}_i$ only depends on the generalized displacements so that

$$\frac{d}{dt}\left(\frac{\partial \mathbf{r}_i}{\partial q_r}\right) = \sum_{k=1}^{n} \frac{\partial^2 \mathbf{r}_i}{\partial q_r \partial q_k}\dot{q}_k + \frac{\partial^2 \mathbf{r}_i}{\partial q_r \partial t}$$

Next the velocity vector is calculated:

$$\dot{\mathbf{r}}_i = \sum_{k=1}^{n} \frac{\partial \mathbf{r}_i}{\partial q_k} \dot{q}_k + \frac{\partial \mathbf{r}_i}{\partial t}$$

and then its partial derivative with respect to the generalized coordinates is calculated:

$$\frac{\partial \dot{\mathbf{r}}_i}{\partial q_r} = \sum_{k=1}^{n} \frac{\partial^2 \mathbf{r}_i}{\partial q_k \partial q_r} \dot{q}_k + \frac{\partial^2 \mathbf{r}_i}{\partial t \partial q_r}$$

If $\mathbf{r}_i$ is twice differentiable, then this equation is the same as the equation for $\frac{d}{dt}\left(\frac{\partial \mathbf{r}_i}{\partial q_r}\right)$.

Therefore, the second expression on the right hand side of this equation

$$\ddot{\mathbf{r}}_i \cdot \frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_r} = \frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_r}\left(\frac{1}{2}\dot{\mathbf{r}}_i \bullet \dot{\mathbf{r}}_i\right)\right) - \dot{\mathbf{r}}_i \bullet \frac{d}{dt}\left(\frac{\partial \mathbf{r}_i}{\partial q_r}\right)$$

can be written as

$$\ddot{\mathbf{r}}_i \cdot \frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_r} = \frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_r}\left(\frac{1}{2}\dot{\mathbf{r}}_i \bullet \dot{\mathbf{r}}_i\right)\right) - \dot{\mathbf{r}}_i \bullet \frac{\partial \dot{\mathbf{r}}_i}{\partial q_r}$$

and then the chain rule is applied to the last expression to obtain the equation that we wished to prove:

$$\ddot{\mathbf{r}}_i \cdot \frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_r} = \frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_r}\left(\frac{1}{2}\dot{\mathbf{r}}_i \bullet \dot{\mathbf{r}}_i\right)\right) - \frac{\partial}{\partial q_r}\left(\frac{1}{2}\dot{\mathbf{r}}_i \bullet \dot{\mathbf{r}}_i\right)$$