ME 563
Mechanical Vibrations
Lecture #9

Second and Fourth Order Continuous Differential Equations

Continuous Systems

If mechanical systems have continuously distributed mass and elasticity (strings, membranes, beams, plates, shells, etc.), then it is necessary to use differential equations of motion that take into account the continuity in the system.

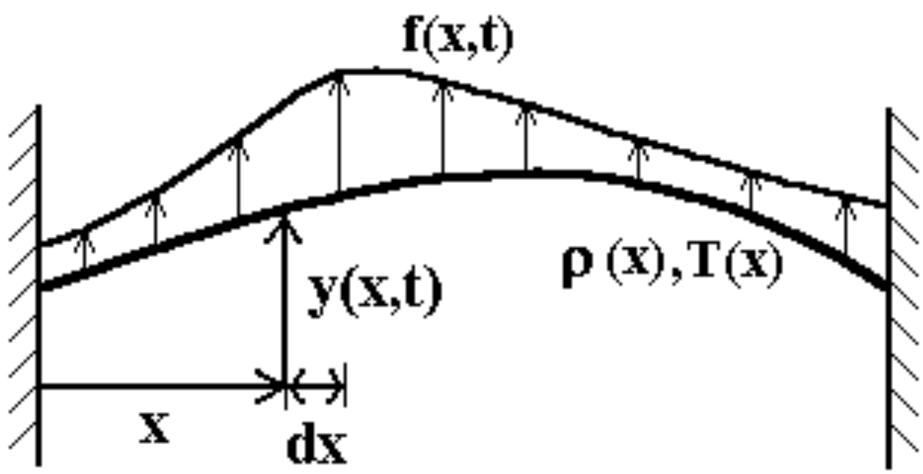
Assume that:

- **Material properties are continuous (density, modulus, etc.)**
- **Small motions meaning geometric nonlinearities are absent**
- **Linear elastic behavior of the material**

String Example

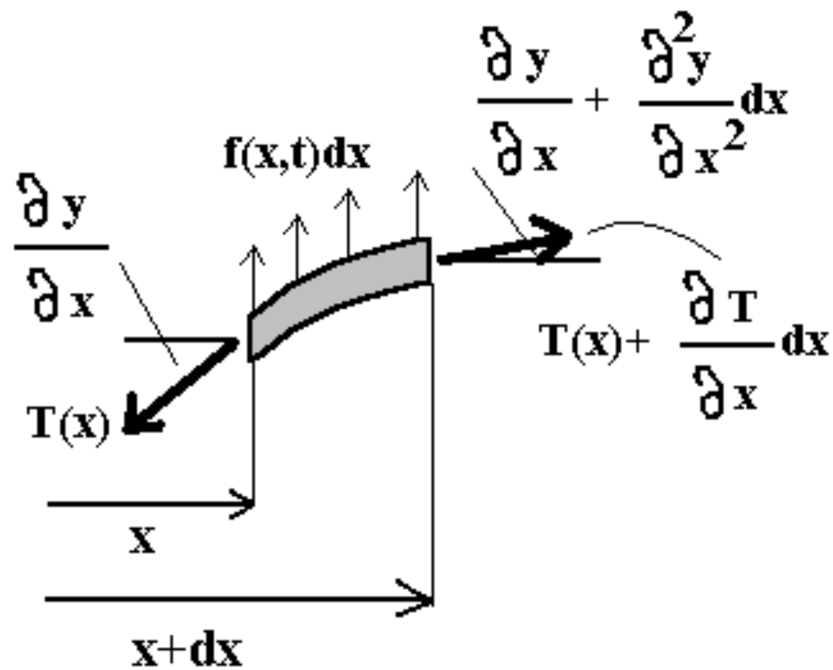
Consider the string shown below subjected to a distributed force that varies with time, f(x,t).

The string displaces in the transverse (vertical) direction at a position x from the left end by y(x,t) as a function of time.



String Example

The free body diagram is drawn using the assumption that tension, density, force, displacement, and slope are continuously varying functions.



String Example

Only the first two terms of the Taylor series expansion for the functions are used (these lead to a linear equation of motion).

$$dM\ddot{y} = +\uparrow \sum_{dM} F$$

$$\left(\rho(x)dx\right)\frac{\partial^2 y(x,t)}{\partial t^2} = \left(T(x) + \frac{\partial T(x)}{\partial x}dx\right)\left(\frac{\partial y(x,t)}{\partial x} + \frac{\partial^2 y}{\partial x^2}dx\right) - T(x)\frac{\partial y(x,t)}{\partial x} + f(x,t)dx$$

$$\rho(x)\frac{\partial^2 y(x,t)}{\partial t^2}dx = \frac{\partial T(x)}{\partial x}\frac{\partial y(x,t)}{\partial x}dx + T(x)\frac{\partial^2 y}{\partial x^2}dx + \frac{\partial T(x)}{\partial x}\frac{\partial^2 y}{\partial x^2}(dx)^2 + f(x,t)dx$$

$$\rho(x)\frac{\partial^2 y(x,t)}{\partial t^2} = \frac{\partial}{\partial x}\left(T(x)\frac{\partial y(x,t)}{\partial x}\right) + f(x,t) + O(dx)$$

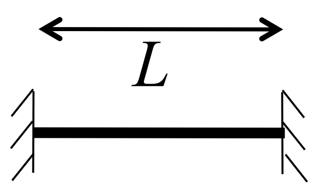
Stiffness (restoring force)

Can we ignore this?

String Example

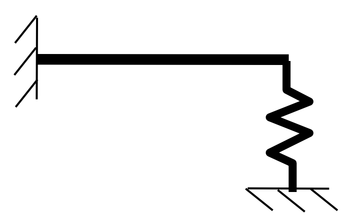
$$\rho(x)\frac{\partial^2 y(x,t)}{\partial t^2} = \frac{\partial}{\partial x}\left(T(x)\frac{\partial y(x,t)}{\partial x}\right) + f(x,t)$$

This equation of motion is second order in y and t; therefore, two boundary conditions and two initial conditions are required to solve the equation. Consider the B.C.s:



$$y(0,t) = 0 = y(L,t)$$

Geometric boundary condition

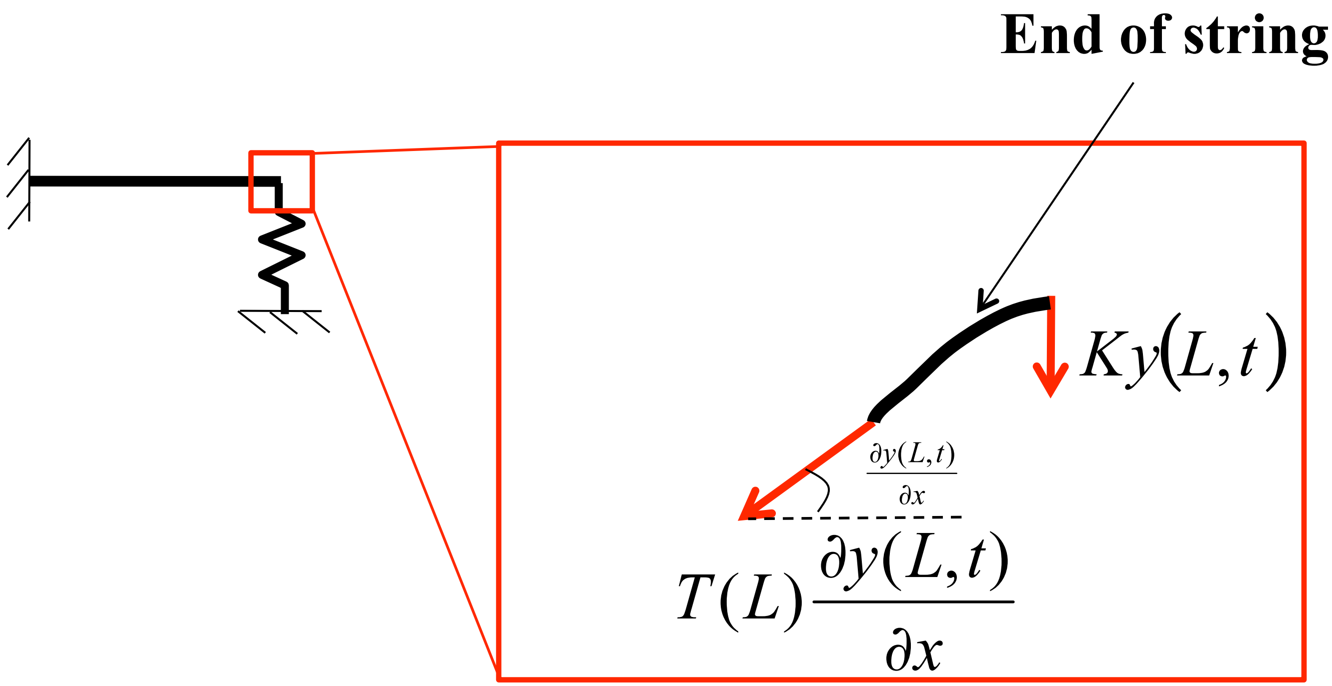


$$y(0,t) = 0, y(L,t) = ?$$

Natural boundary condition

String Example

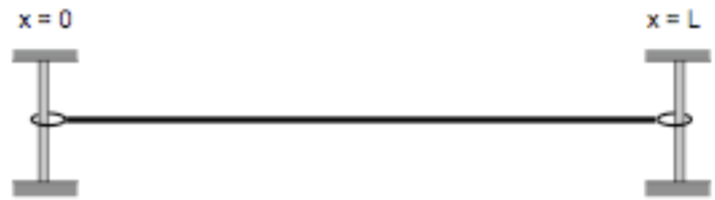
To determine the equation for a natural boundary condition, we usually use Newton-Euler laws to balance dynamic forces.



$$-Ky(L,t)-T(L)\frac{\partial y(L,t)}{\partial x}=0$$

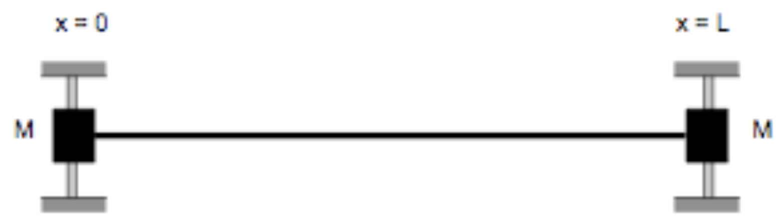
Other BCs (C. Krousgrill)

Free end: massless ring slides on smooth guide ("free" in transverse direction, yet maintains tension in string)



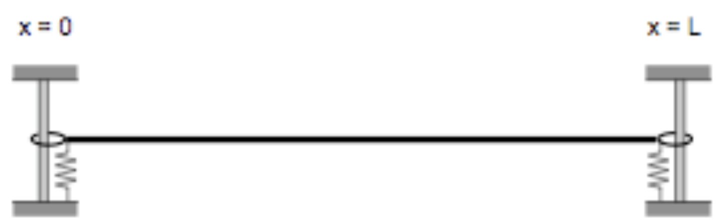
$$\frac{\partial y}{\partial x}(0,t)=0 \qquad \frac{\partial y}{\partial x}(L,t)=0$$

Mass end: block of mass M on smooth guide



$$T\frac{\partial y}{\partial x}(0,t)=M\frac{\partial^2 y}{\partial t^2}(0,t) \qquad T\frac{\partial y}{\partial x}(L,t)=-M\frac{\partial^2 y}{\partial t^2}(L,t)$$

Spring end: spring of stiffness K resists transverse motion on end

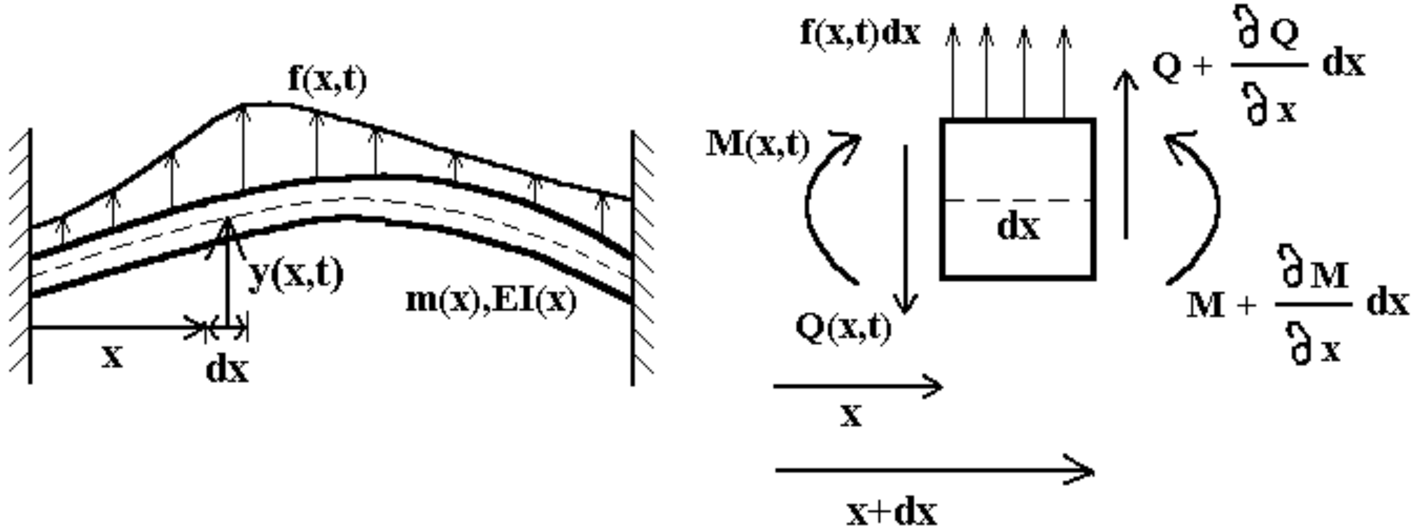


$$T\frac{\partial y}{\partial x}(0,t)=Ky(0,t) \qquad T\frac{\partial y}{\partial x}(L,t)=-Ky(L,t)$$

Beam Example

Consider the beam shown below subjected to a distributed force that varies with time, f(x,t).

The beam displaces in the transverse (vertical) direction at a position x from the left end by y(x,t) as a function of time.



Beam Example

Only the first two terms of the Taylor series expansion for the functions are used (these lead to a linear equation of motion).

$$\big(m(x)dx\big)\frac{\partial^2 y(x,t)}{\partial t^2} = \left(Q(x,t) + \frac{\partial Q}{\partial x}dx\right) - Q(x,t) + f(x,t)dx$$

$$0 = \left(M(x,t) + \frac{\partial M}{\partial x}dx\right) - M(x,t) + \left(Q(x,t) + \frac{\partial Q}{\partial x}dx\right)dx + f(x,t)dx\frac{dx}{2}$$

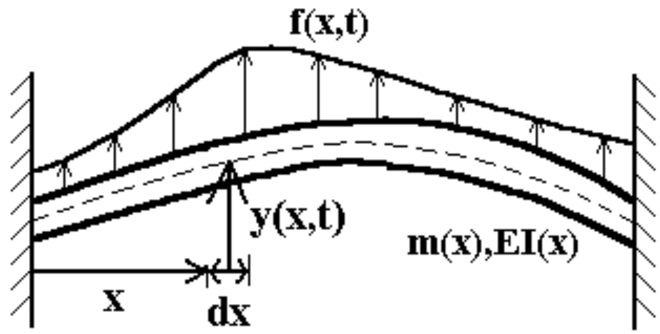
$$m(x)\frac{\partial^2 y(x,t)}{\partial t^2} = -\frac{\partial^2 M}{\partial x^2} + f(x,t)$$

$$M(x,t) = EI(x)\frac{\partial^2 y(x,t)}{\partial x^2}$$

Mass (inertia force)

Stiffness (restoring force)

Beam Example



This equation of motion is fourth order in y; therefore, four boundary conditions are required to solve the equation:

Fixed-fixed

$$y(0,t)=0=\left.\frac{\partial y(x,t)}{\partial x}\right|_{x=0}$$

$$y(L,t)=0=\left.\frac{\partial y(x,t)}{\partial x}\right|_{x=L}$$

Free-free

$$\left.\frac{\partial^2 y(x,t)}{\partial x^2}\right|_{x=0}=0=\left.\frac{\partial^3 y(x,t)}{\partial x^3}\right|_{x=0}$$

$$\left.\frac{\partial^2 y(x,t)}{\partial x^2}\right|_{x=L}=0=\left.\frac{\partial^3 y(x,t)}{\partial x^3}\right|_{x=L}$$

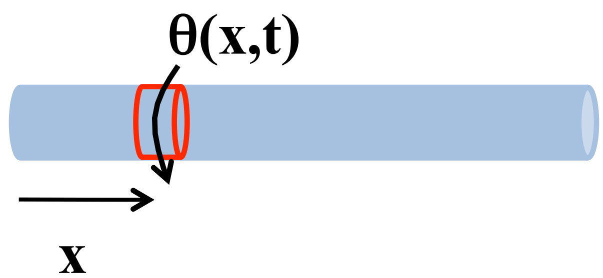
Fixed-hinged

$$y(0,t)=0=\left.\frac{\partial y(x,t)}{\partial x}\right|_{x=0}$$

$$y(L,t)=0=\left.\frac{\partial^2 y(x,t)}{\partial x^2}\right|_{x=L}$$

Torsional Vibration

The shaft displaces in the rotational direction by an amount θ(x,t) at a position x from the left end as a function of time.



$$I_s(x)\frac{\partial^2\theta(x,t)}{\partial t^2}=\frac{\partial}{\partial x}\left(GJ(x)\frac{\partial\theta(x,t)}{\partial x}\right)+\tau(x,t)$$

$I_s(x)$ = Mass polar moment of inertia per unit length
$\tau(x,t)$ = Torque per unit length