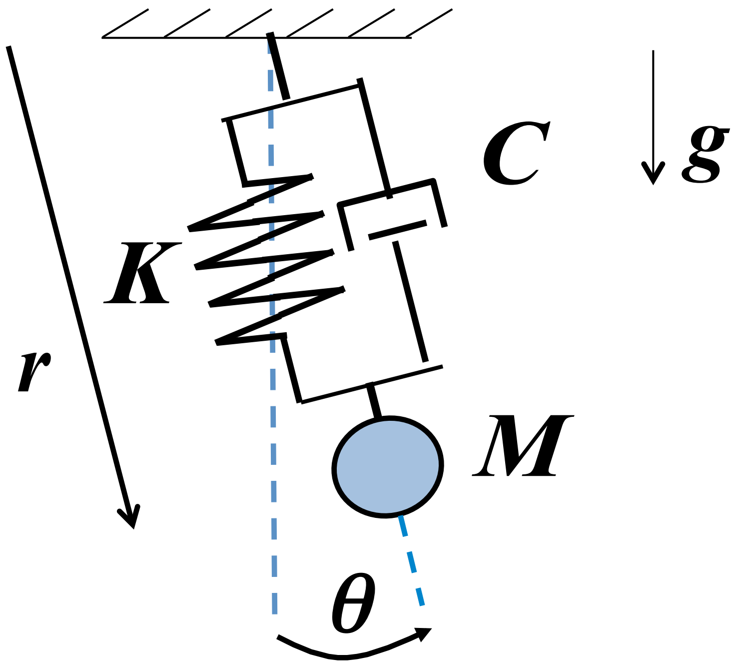
ME 563
Mechanical Vibrations
Lecture #8

Linearizing Equations of Motion
(Direct Method)

Sprung Pendulum



One option for linearizing is to derive the equations of motion FIRST and then:

- Identify equilibria
- Transform coordinates
- Discard higher-order terms

$$M\left(\ddot{r} - r\dot{\theta}^2\right) + K\left(r - r_u\right) - Mg\cos\theta + C\dot{r} = 0$$

$$M\left(2r\dot{r}\dot{\theta} + r^2\ddot{\theta}\right) + Mgr\sin\theta = 0$$

Identify Equilbria

$$\dot{r} = 0 = \ddot{r} \text{ and } \dot{\theta} = 0 = \ddot{\theta}$$

$$Mgr_{eq} \sin\theta_{eq} = 0 \Rightarrow \theta_{eq} = 0, \pm\pi, etc.$$

$$K\left(r_{eq} - r_u\right) - Mg\cos\theta_{eq} = 0 \Rightarrow r_{eq} = r_u \pm \frac{Mg}{K}$$

Makes sense…

a)Pendulum can be straight down or straight up.
b)Spring compresses accordingly.

Transform Coordinates

$$\rightarrow \quad r = r_d + r_{eq} \text{ and } \theta = \theta_d + \theta_{eq}$$

$$r_{eq} = r_u + Mg/K, \theta_{eq} = 0$$

$$M\left(\ddot{r}_d - \left(r_d + r_{eq}\right)\dot{\theta}_d^{\,2}\right) + K\left(r_d + r_{eq} - r_u\right) - Mg\cos\left(\theta_d + \theta_{eq}\right) + C\dot{r}_d = 0$$

$$M\left(\ddot{r}_d - \left(r_d + r_{eq}\right)\dot{\theta}_d^{\,2}\right) + K\left(r_d + r_{eq} - r_u\right) - Mg\left(\cos\theta_d\cos\theta_{eq} - \sin\theta_d\sin\theta_{eq}\right) + C\dot{r}_d = 0$$

$$\rightarrow \quad M\left(\ddot{r}_d - \left(r_d + r_{eq}\right)\dot{\theta}_d^{\,2}\right) + C\dot{r}_d + K\left(r_d + \frac{Mg}{K}\right) - Mg\cos\theta_d = 0$$

$$M\left(2\left(r_d + r_{eq}\right)\dot{r}_d\dot{\theta}_d + \left(r_d + r_{eq}\right)^2\ddot{\theta}_d\right) + Mg\left(r_d + r_{eq}\right)\sin\left(\theta_d + \theta_{eq}\right) = 0$$

$$M\left(2\left(r_d + r_{eq}\right)\dot{r}_d\dot{\theta}_d + \left(r_d + r_{eq}\right)^2\ddot{\theta}_d\right) + Mg\left(r_d + r_{eq}\right)\left(\sin\theta_d\cos\theta_{eq} + \sin\theta_{eq}\cos\theta_d\right) = 0$$

$$\rightarrow \quad M\left(2\left(r_d + r_{eq}\right)\dot{r}_d\dot{\theta}_d + \left(r_d + r_{eq}\right)^2\ddot{\theta}_d\right) + Mg\left(r_d + r_{eq}\right)\sin\theta_d = 0$$

Transform Coordinates

$$\rightarrow M\left(\ddot{r}_d - \left(r_d + r_{eq}\right)\dot{\theta}_d^{\,2}\right) + C\dot{r}_d + K\left(r_d + \frac{Mg}{K}\right) - Mg\cos\theta_d = 0$$

$$\rightarrow M\ddot{r}_d + C\dot{r}_d + K\left(r_d + \frac{Mg}{K}\right) - Mg + O\left(\dot{\theta}_d^{\,2}, \theta_d^{\,2}\right) = 0$$

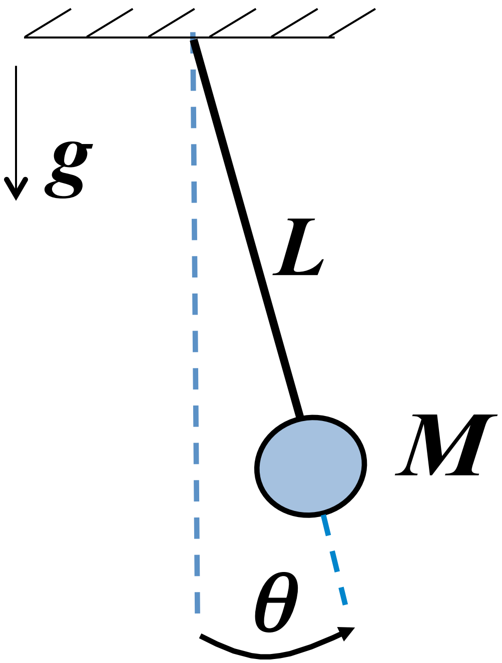
$$\rightarrow M\ddot{r}_d + C\dot{r}_d + Kr_d + O\left(\dot{\theta}_d^{\,2}, \theta_d^{\,2}\right) = 0$$

$$\rightarrow M\left(2\left(r_d + r_{eq}\right)\dot{r}_d\dot{\theta}_d + \left(r_d + r_{eq}\right)^2\ddot{\theta}_d\right) + Mg\left(r_d + r_{eq}\right)\sin\theta_d = 0$$

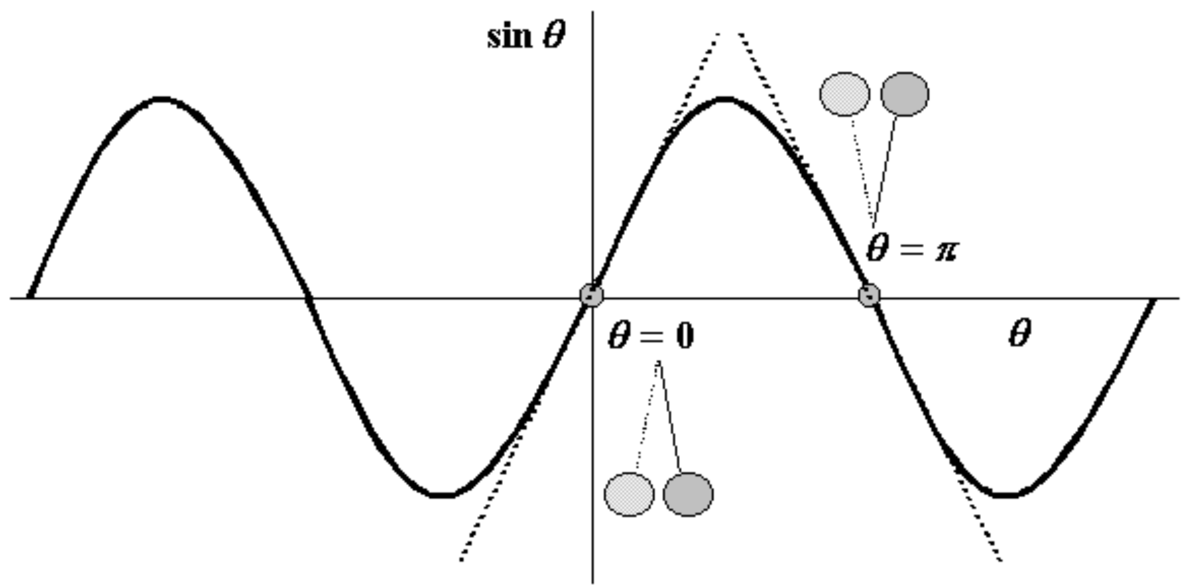
$$\rightarrow Mr_{eq}^2\ddot{\theta}_d + Mgr_{eq}\theta_d + O\left(\dot{r}_d\dot{\theta}_d, r_d\ddot{\theta}_d, r_d^2\ddot{\theta}_d, \theta_d^{\,3}\right) = 0$$

Physical Meaning of Linearization (Pendulum)

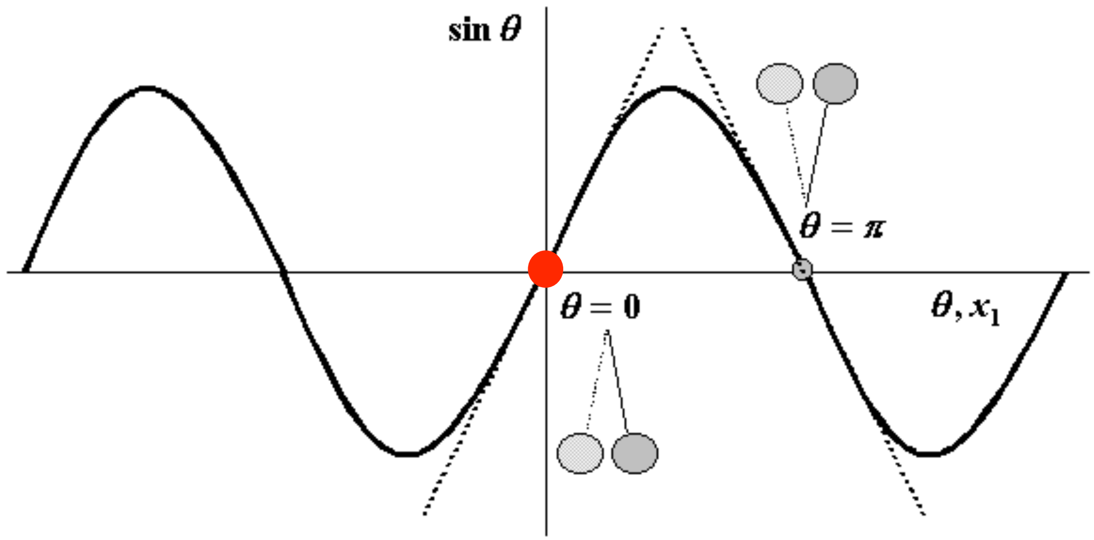
$$ML^2\ddot{\theta} + MgL\sin\theta = 0$$



$$\sin\theta = \sin\theta_o + \sum_{n=1}^{\infty}\frac{1}{n!}\left[\frac{d^{(n)}}{d\theta^{(n)}}(\sin\theta)\right]\Bigg|_{\theta=\theta_o}(\theta-\theta_o)^n = \theta - \frac{1}{6}\theta^3 + \cdots$$



Simple Pendulum

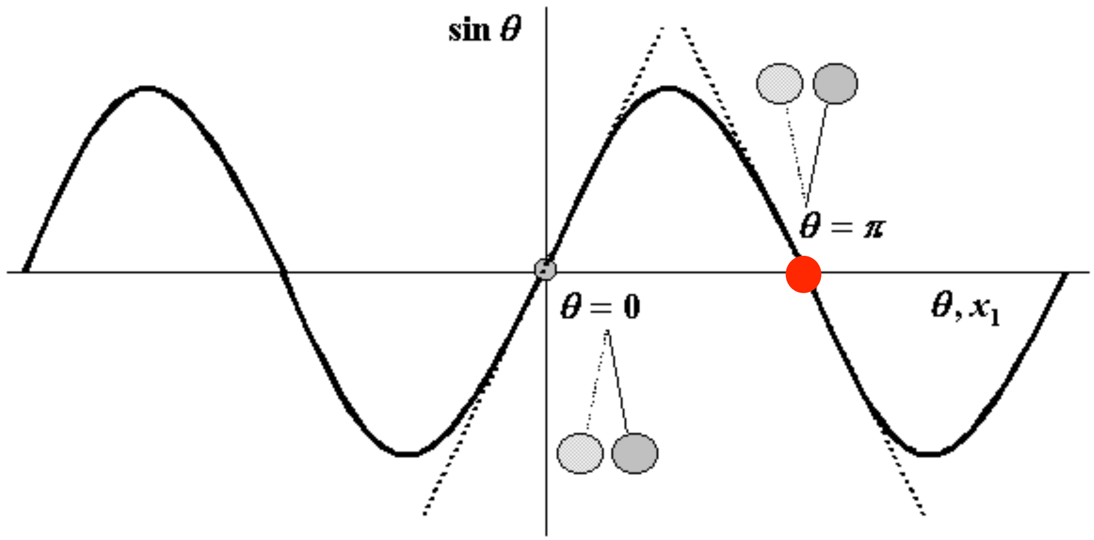


$$ML^2\ddot{\theta} + MgL\sin\theta = 0$$

$$\theta_{eq} = 0 \longrightarrow ML^2\ddot{\theta}_d + MgL\sin(\theta_d + 0) = 0$$

$$\ddot{\theta}_d + \frac{g}{L}\theta_d \cong 0$$

Simple Pendulum



$$ML^2\ddot{\theta} + MgL\sin\theta = 0$$

$$\theta_{eq} = \pi \longrightarrow ML^2\ddot{\theta} + MgL\sin(\theta_d + \pi) = 0$$

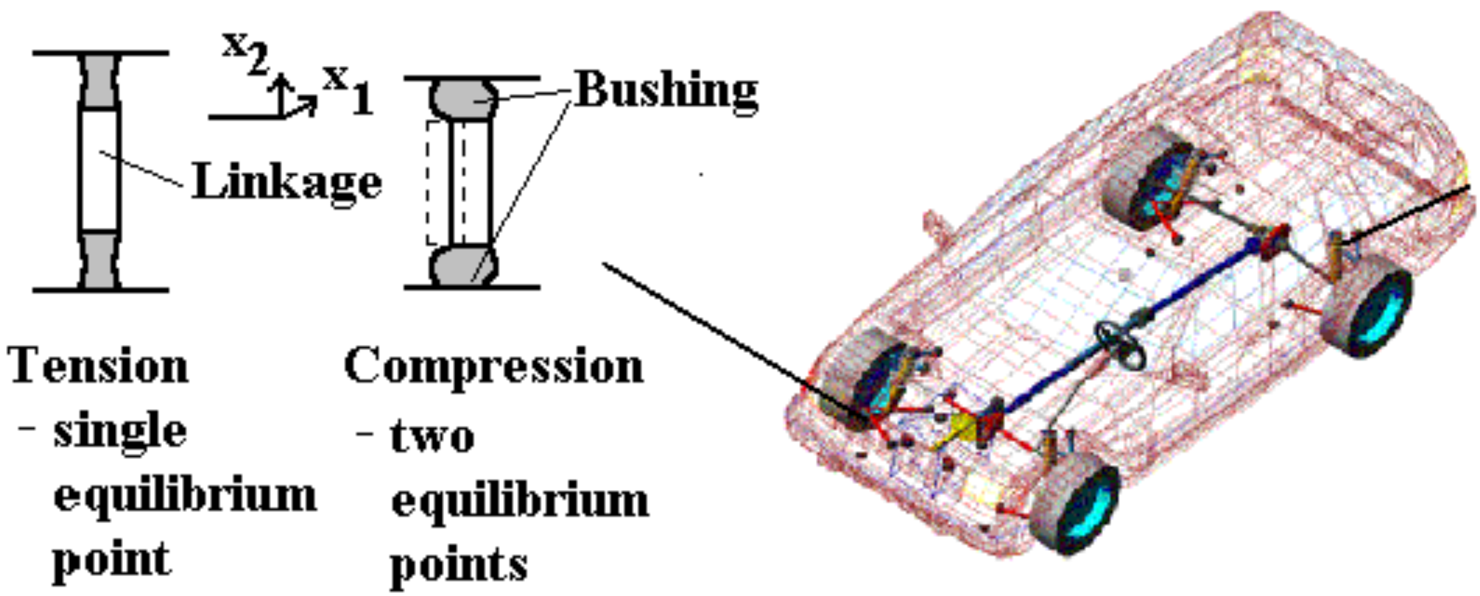
$$ML^2\ddot{\theta} - MgL\sin(\theta_d) = 0$$

$$\ddot{\theta}_d - \frac{g}{L}\theta_d \cong 0$$

Sprung Linkage

Preloaded mechanical components can exhibit multiple equilibrium points, so the vibrations change depending on the equilibrium point.

$$M\ddot{y} + C\dot{y} - Ky + \mu y^3 = 0$$



Sprung Linkage

To find the equilibrium points, we can either:

$$M\ddot{y} + C\dot{y} - Ky + \mu y^3 = 0$$

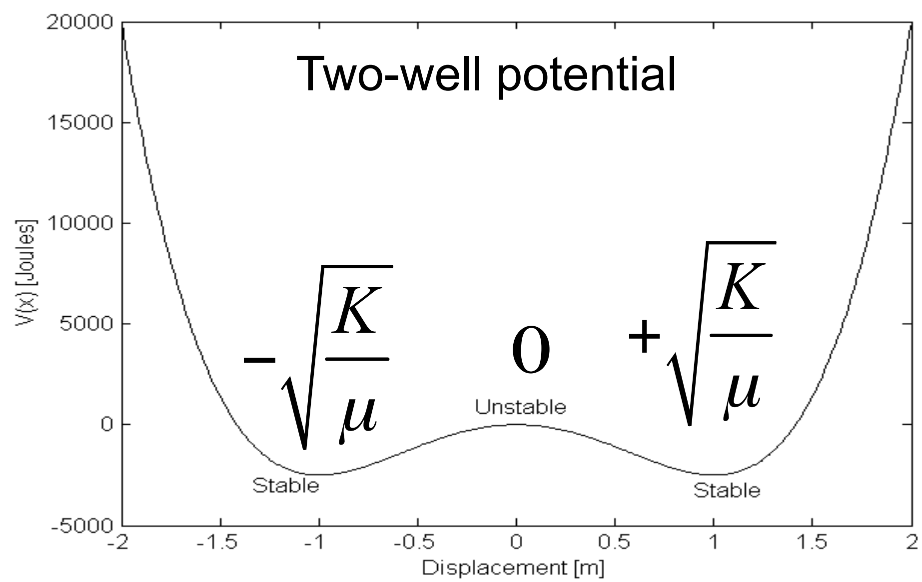
- Set the rate of change in the states to zero.

$$-Ky + \mu y^3 = 0 \Rightarrow y = 0, \pm\sqrt{K/\mu}$$

- Set the derivative of the potential energy to zero.

$$V(y) = -\frac{K}{2}y^2 + \frac{\mu}{4}y^4$$

$$V'(y) = -Ky + \mu y^3 = 0$$



Sprung Linkage

Then the procedure described previously is followed to transform the coordinate and linearize the equation of motion.

→ $y = y_d + \sqrt{K/\mu}$

→ $M\ddot{y}_d + C\dot{y}_d - K\left(y_d + \sqrt{K/\mu}\right) + \mu\left(y_d + \sqrt{K/\mu}\right)^3 = 0$

→ $M\ddot{y}_d + C\dot{y}_d - K\left(y_d + \sqrt{K/\mu}\right) + \mu\left(y_d^3 + 3y_d^2\sqrt{K/\mu} + 3y_d K/\mu + (K/\mu)^{3/2}\right) = 0$

→ $M\ddot{y}_d + C\dot{y}_d + 2Ky_d + \mu(K/\mu)^{3/2} + O\left(y_d^3, y_d^2,\right) = 0$

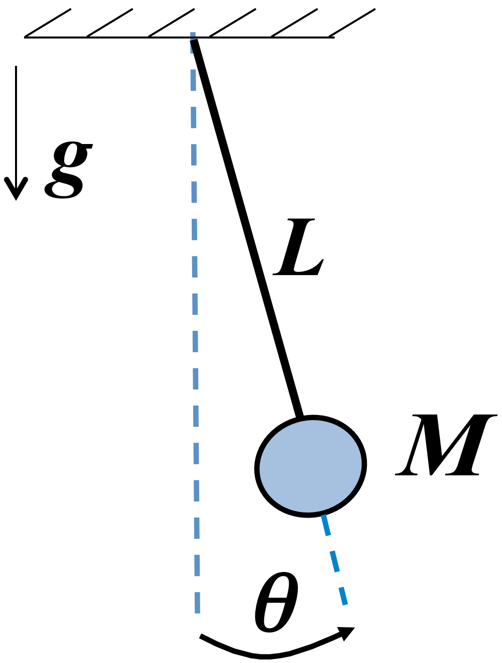
→ $y = y_d + 0$

→ $M\ddot{y}_d + C\dot{y}_d - K\left(y_d\right) + \mu\left(y_d\right)^3 = 0$

→ $M\ddot{y}_d + C\dot{y}_d - Ky_d - K\sqrt{K/\mu} + O(y_d^3) = 0$

Lagrange's Approach

To linearize the equations of motion, it is also possible to eliminate higher order terms in L = T – V.



$$T = \frac{1}{2} M \left(L\dot{\theta} \right)^2$$

$$V = -MgL\cos\theta$$

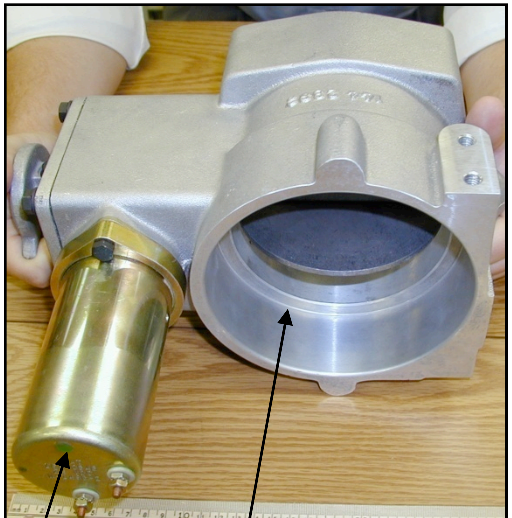
$$= -MgL\left[1 - \frac{\theta^2}{2!} + O\left(\theta^4\right) \right]$$

Keep quadratic terms only.

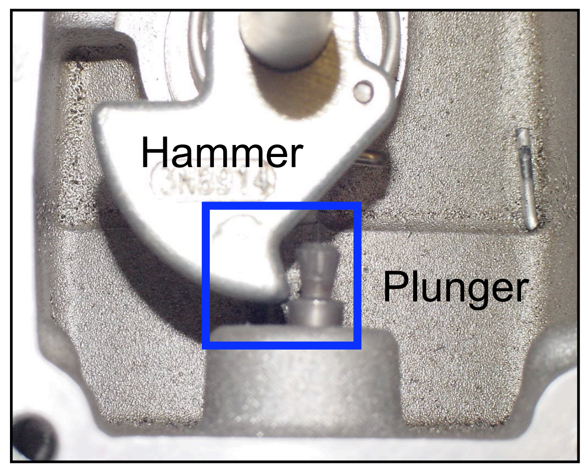
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = Q_\theta^*$$

$$ML^2\ddot{\theta} + MgL\theta = 0$$

Example of Multiple Equilibrium Points



After operation in the field



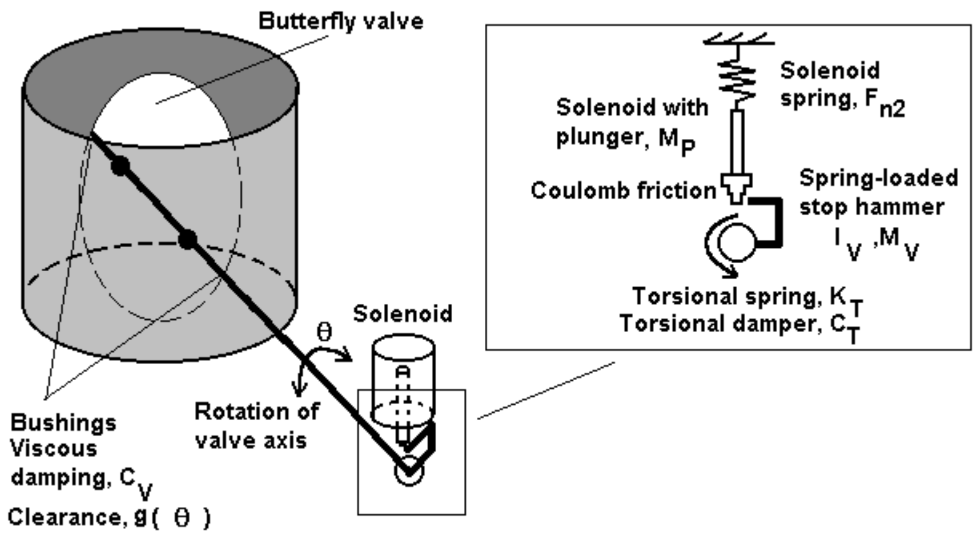
Vibrations about one equilibrium point lead to more damage in the bushing than for the other point.

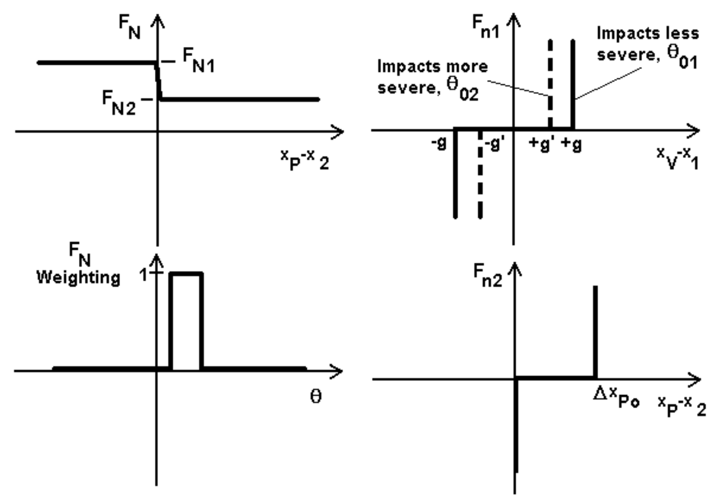
Engine Shutoff Valve

$$M_V \ddot{x}_V + C_V(\dot{x}_V - \dot{x}_1) + F_{n1}(x_V - x_1, g(\theta)) = 0$$

$$I_V \ddot{\theta} + C_T \dot{\theta} + K_T \theta - F_N \cdot c \cos\theta = 0$$

$$M_P \ddot{x}_P + C_P(\dot{x}_P - \dot{x}_2) + F_{n2}(x_P - x_2) + F_T = 0$$





Engine Shutoff Valve

When simulations are carried out for various initial conditions, it is clear that the impacts can be severe for long periods of time whenever the hammer 'hops' up onto the lip of the plunger.