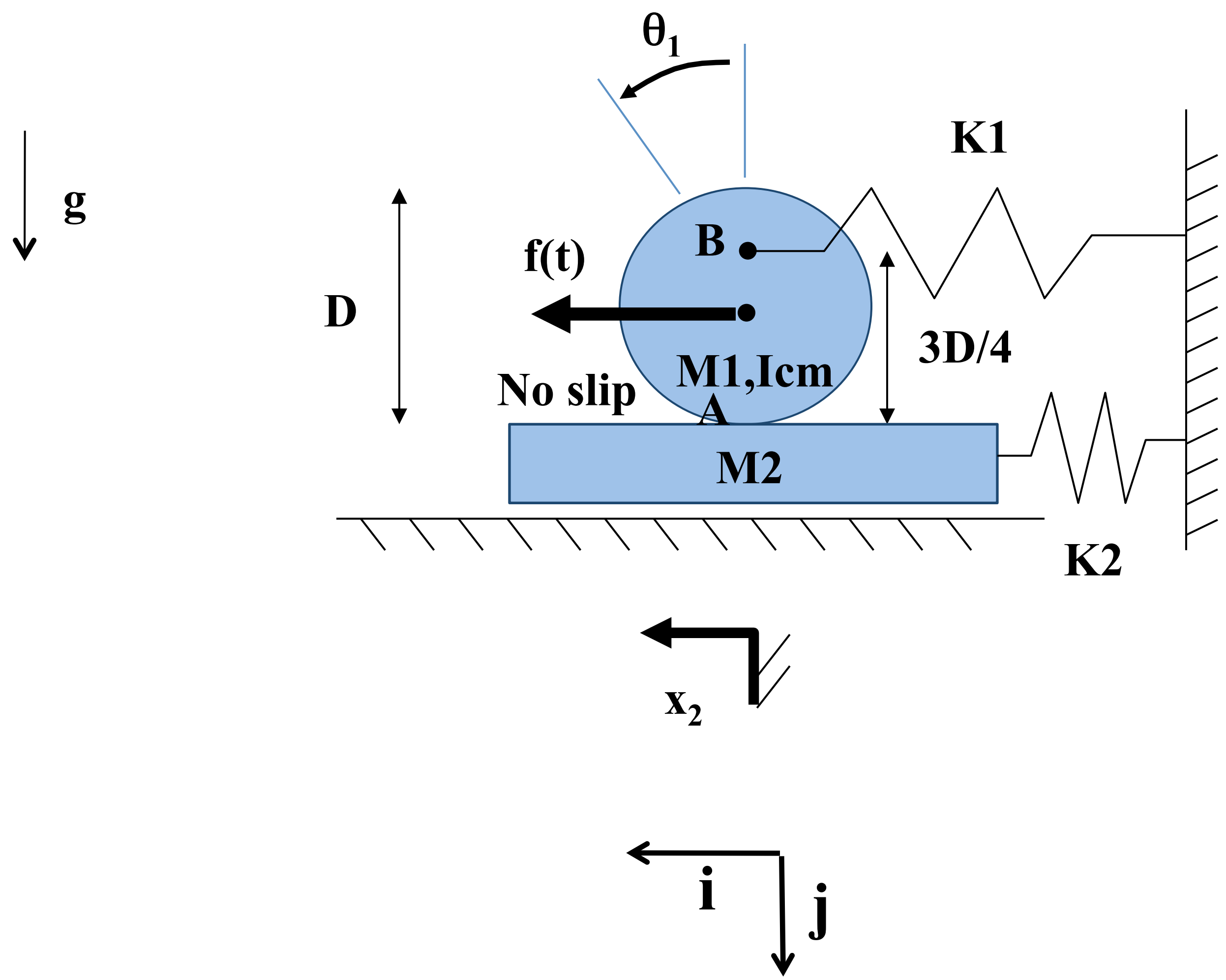
ME 563
Mechanical Vibrations
Lecture #6

Lagrange's Method for
Deriving Equations of Motion
(examples)

Example
Rolling Disc on a Moving Cart



Example
Rolling Disc on a Moving Cart

→ $$T = \frac{1}{2} M_1 \dot{x}_1^{\;2} + \frac{1}{2} I_{cm} \dot{\theta}_1^2 + \frac{1}{2} M_2 \dot{x}_2^{\;2}$$

→ $$V = \frac{1}{2} K_1 x_B^{\;2} + \frac{1}{2} K_2 x_2^{\;2}$$

↓ **Eliminate constraints (keep x_2 and θ_1)**

→ $$T = \frac{1}{2} M_1 \left(\dot{x}_2 + \frac{D}{2} \dot{\theta}_1 \right)^2 + \frac{1}{2} I_{cm} \dot{\theta}_1^2 + \frac{1}{2} M_2 \dot{x}_2^{\;2}$$

→ $$V = \frac{1}{2} K_1 \left(x_2 + \frac{3D}{4} \theta_1 \right)^2 + \frac{1}{2} K_2 x_2^{\;2}$$

Example
Rolling Disc on a Moving Cart

$$Q^*_{\theta_1} = \sum_{i=1}^{N} \left(\mathbf{F}_{i,nc}\right) \cdot \frac{\partial \mathbf{r}_i}{\partial \theta_1}$$

$$= \sum_{i=1}^{N} f(t)\mathbf{i} \cdot \frac{\partial\left(\left(x_2 + \frac{D}{2}\theta_1\right)\mathbf{i}\right)}{\partial \theta_1}$$

$$= f(t)\mathbf{i} \cdot \frac{D}{2}\mathbf{i}$$

$$= \frac{D}{2} f(t)$$

$$Q^*_{x_2} = \sum_{i=1}^{N} \left(\mathbf{F}_{i,nc}\right) \cdot \frac{\partial \mathbf{r}_i}{\partial x_2}$$

$$= \sum_{i=1}^{N} f(t)\mathbf{i} \cdot \frac{\partial\left(\left(x_2 + \frac{D}{2}\theta_1\right)\mathbf{i}\right)}{\partial x_2}$$

$$= f(t)\mathbf{i} \cdot 1\mathbf{i}$$

$$= f(t)$$

Example
Rolling Disc on a Moving Cart

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_r} - \frac{\partial T}{\partial q_r} + \frac{\partial V}{\partial q_r} - Q_r^*$$

$$\theta_1: \frac{d}{dt}\left(\frac{D}{2}M_1\left(\dot{x}_2 + \frac{D}{2}\dot{\theta}_1\right) + I_{cm}\dot{\theta}_1\right) - 0 + \frac{3D}{4}K_1\left(x_2 + \frac{3D}{4}\theta_1\right) - \frac{D}{2}f(t) = 0$$

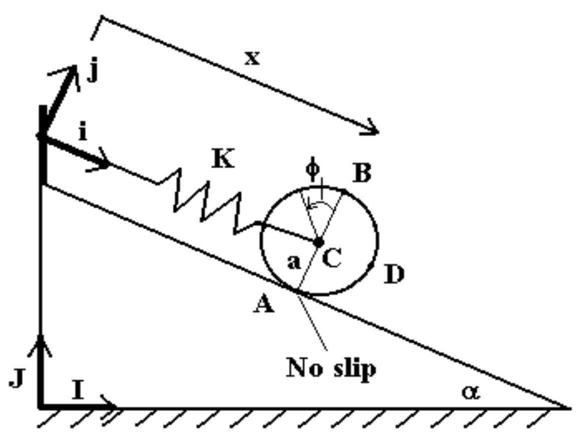
$$\frac{D}{2}M_1\left(\ddot{x}_2 + \frac{D}{2}\ddot{\theta}_1\right) + I_{cm}\ddot{\theta}_1 + \frac{3D}{4}K_1\left(x_2 + \frac{3D}{4}\theta_1\right) = \frac{D}{2}f(t)$$

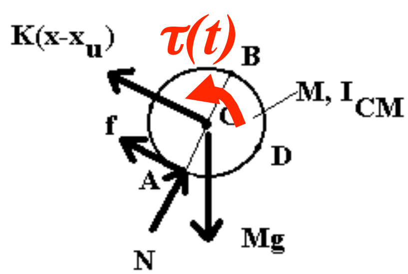
$$x_2: \frac{d}{dt}\left(M_1\left(\dot{x}_2 + \frac{D}{2}\dot{\theta}_1\right) + M_2\dot{x}_2\right) - 0 + K_1\left(x_2 + \frac{3D}{4}\theta_1\right) + K_2x_2 - f(t) = 0$$

$$M_1\left(\ddot{x}_2 + \frac{D}{2}\ddot{\theta}_1\right) + M_2\ddot{x}_2 + K_1\left(x_2 + \frac{3D}{4}\theta_1\right) + K_2x_2 = f(t)$$

Torque Example
Rolling Disc on Incline

In the last slide, we considered forces but what about a torque applied at the center of mass of the rolling disc? How do we compute the generalized "force."





Torque, *τ(t)*, acting <u>about</u> point C

We need to think in terms of torque acting about a point

Explanation
Rolling Disc on Incline

There are several things to keep in mind with a torque:

-A torque is not a force; therefore, the generalized coordinate we need to use is ϕ not x.
-Our intuition about torques and where they are applied can let us down; when we apply a torque, we say it is applied <u>about</u> a point in the plane, not <u>at</u> a point x.
-To understand the solution, we can think in terms of the work done by the generalized "forces"; the work done by these forces must be equal to the actual work done by the forces and torques applied to the body.

Option #1
Rolling Disc on Incline

So the work done by a force *p(t)* applied <u>at</u> c is:

$$dW = p(t)\mathbf{i}\cdot d\mathbf{r}_c = p(t)\mathbf{i}\cdot dx\mathbf{i} = \underline{p(t)dx = Q_x^* dx}$$

And the work done by a torque *τ(t)* applied <u>about</u> c is:

$$dW = \tau(t)\mathbf{k}\cdot d\boldsymbol{\theta}_c = \tau(t)\mathbf{k}\cdot d\phi\mathbf{k} = \underline{\tau(t)d\phi = Q_\phi^* d\phi}$$

Then we can find the generalized "force" by equating the last two expressions in the these two equations.

Option #2
Rolling Disc on Incline

Or, we can solve for the generalized "force" by applying the equation we used in all the other examples we have done:

Force *p(t)* <u>at</u> c, $\mathbf{r}_C = x\mathbf{i}$

$$Q_r^* = \sum_{i=1}^{N} \left(\mathbf{F}_{i,nc}\right) \cdot \frac{\partial \mathbf{r}_i^{(k)}}{\partial q_r^{(k)}}$$

$$Q_x^* = \sum_{i=1}^{N} \left(p(t)\right)\mathbf{i} \cdot \frac{\partial \mathbf{r}_C}{\partial x}$$

$$= p(t)\mathbf{i} \cdot \mathbf{i}$$

$$= p(t)$$

Torque *τ(t)* <u>about</u> c, $\boldsymbol{\theta}_C = \phi\mathbf{k}$

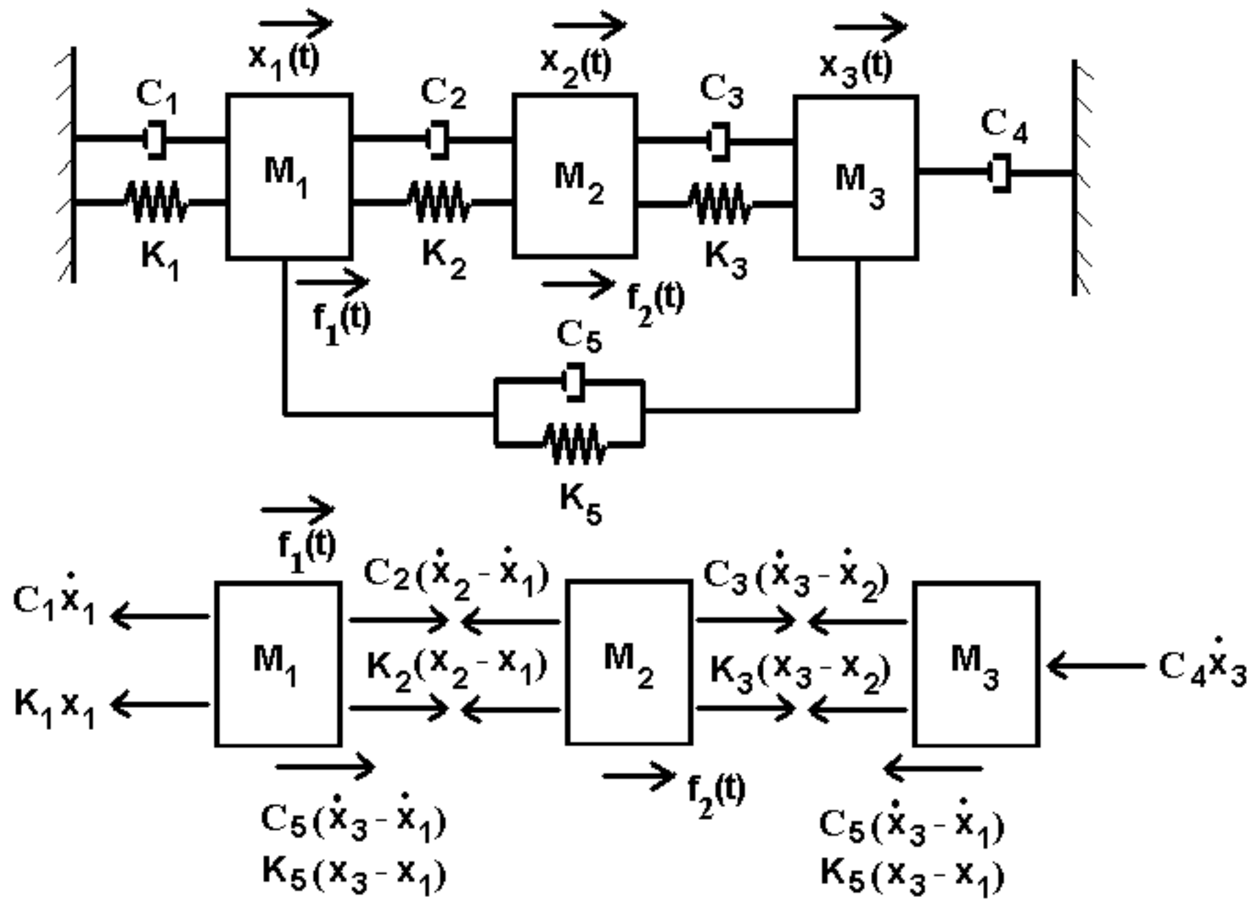
$$Q_r^* = \sum_{i=1}^{N} \left(\mathbf{F}_{i,nc}\right) \cdot \frac{\partial \mathbf{r}_i^{(k)}}{\partial q_r^{(k)}}$$

$$Q_\phi^* = \sum_{i=1}^{N} \left(\tau(t)\right)\mathbf{k} \cdot \frac{\partial \boldsymbol{\theta}_C}{\partial \phi}$$

$$= \tau(t)\mathbf{k} \cdot \mathbf{k}$$

$$= \tau(t)$$

Example
Two Degree of Freedom System



Example
Two Degree of Freedom System

$$T = \frac{1}{2}M_1\dot{x}_1^2 + \frac{1}{2}M_2\dot{x}_2^2 + \frac{1}{2}M_3\dot{x}_3^2$$

$$V = \frac{1}{2}K_1x_1^2 + \frac{1}{2}K_2(x_2 - x_1)^2 + \frac{1}{2}K_3(x_3 - x_2)^2 + \frac{1}{2}K_5(x_3 - x_1)^2$$

Raleigh's dissipation function

$$\Re = \frac{1}{2}C_1\dot{x}_1^2 + \frac{1}{2}C_2(\dot{x}_2 - \dot{x}_1)^2 + \frac{1}{2}C_3(\dot{x}_3 - \dot{x}_2)^2 + \frac{1}{2}C_5(\dot{x}_3 - \dot{x}_1)^2 + \frac{1}{2}C_4\dot{x}_3^2$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_r} - \frac{\partial L}{\partial q_r} + \frac{\partial \Re}{\partial \dot{q}_r} = Q_r^*$$

Example
Two Degree of Freedom System

$$\begin{bmatrix} M_1 & 0 & 0 \\ 0 & M_2 & 0 \\ 0 & 0 & M_3 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + \begin{bmatrix} C_1 + C_2 + C_5 & -C_2 & -C_5 \\ -C_2 & C_2 + C_3 & -C_3 \\ -C_5 & -C_3 & C_5 + C_3 + C_4 \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{Bmatrix} +$$

$$\begin{bmatrix} K_1 + K_2 + K_5 & -K_2 & -K_5 \\ -K_2 & K_2 + K_3 & -K_3 \\ -K_5 & -K_3 & K_5 + K_3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} f_1(t) \\ f_2(t) \\ 0 \end{Bmatrix}$$

Check for accuracy through symmetry and positive definiteness of element matrices.

Example
Two Degree of Freedom System

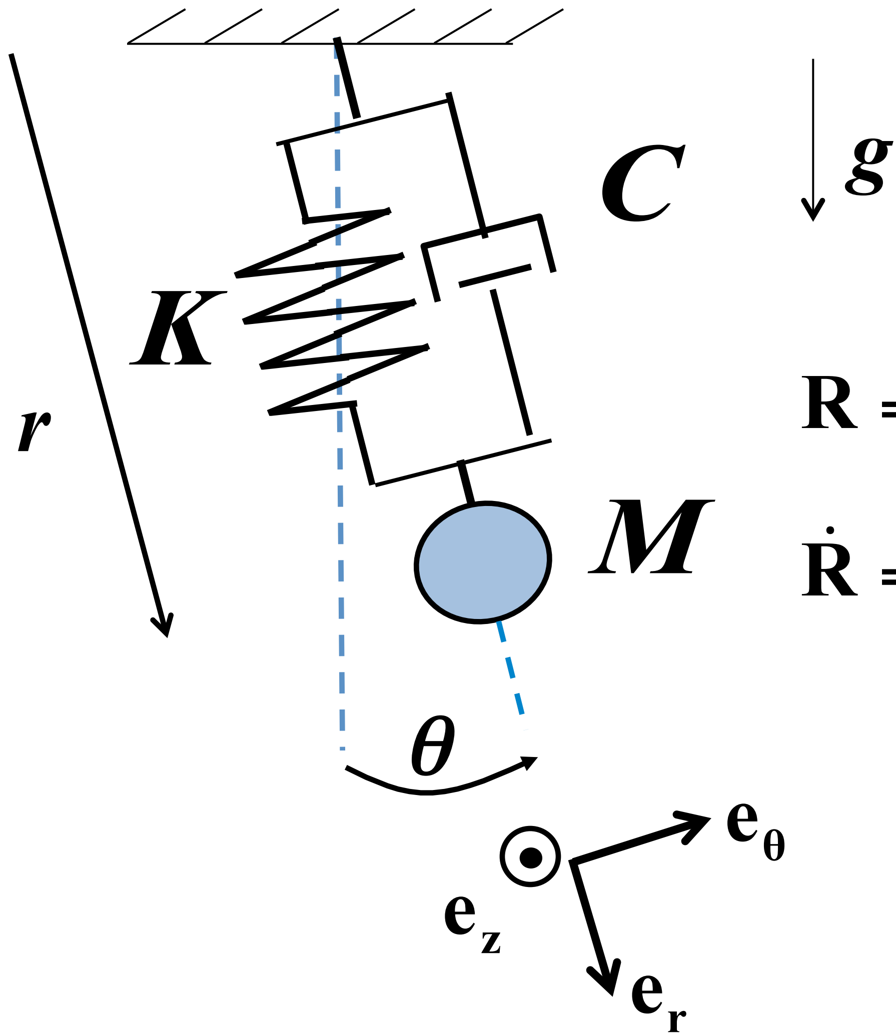
$$\begin{bmatrix} M_1 & 0 & 0 \\ M_1 & M_2 & 0 \\ 0 & 0 & M_3 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2^* \\ \ddot{x}_3 \end{Bmatrix} + \begin{bmatrix} C_1 + C_5 & -C_2 & -C_5 \\ C_3 & C_2 + C_3 & -C_3 \\ -C_5 - C_3 & -C_3 & C_5 + C_3 + C_4 \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2^* \\ \dot{x}_3 \end{Bmatrix} +$$

$$\begin{bmatrix} K_1 + K_2 + K_5 & -K_2 & -K_5 \\ K_3 & K_2 + K_3 & -K_3 \\ -K_5 - K_3 & -K_3 & K_5 + K_3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2^* \\ x_3 \end{Bmatrix} = \begin{Bmatrix} f_1(t) \\ f_2(t) \\ 0 \end{Bmatrix}$$

$$x_2^* = x_2 - x_1$$

Use of relative coordinates will distort symmetry.

Sprung Pendulum



$$\mathbf{R} = r\mathbf{e_r}$$

$$\dot{\mathbf{R}} = \dot{r}\mathbf{e_r} + r\frac{d\mathbf{e_r}}{dt} = \dot{r}\mathbf{e_r} + r\dot{\theta}\mathbf{e_\theta}$$

Sprung Pendulum

Rate of change of unit vector

$$\frac{d\mathbf{e_r}}{dt} = \vec{\omega} \times \mathbf{e_r} = \dot{\theta}\mathbf{e_z} \times \mathbf{e_r}$$

$$= \dot{\theta}\mathbf{e_\theta}$$

→ $$\dot{\mathbf{R}} = \dot{r}\mathbf{e_r} + r\frac{d\mathbf{e_r}}{dt} = \dot{r}\mathbf{e_r} + r\dot{\theta}\mathbf{e_\theta}$$

→ $$T = \frac{1}{2}M\dot{\mathbf{R}} \cdot \dot{\mathbf{R}}$$

→ $$= \frac{1}{2}M\left(\dot{r}\mathbf{e_r} + r\dot{\theta}\mathbf{e_\theta}\right) \cdot \left(\dot{r}\mathbf{e_r} + r\dot{\theta}\mathbf{e_\theta}\right)$$

→ $$= \frac{1}{2}M\left(\dot{r}^2 + 0 + r^2\dot{\theta}^2\right)$$

→ $$V = \frac{1}{2}K\left(r - r_u\right)^2 - Mgr\cos\theta$$

→ $$\Re = \frac{1}{2}C\dot{r}^2$$

Sprung Pendulum

→ $$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_r} - \frac{\partial L}{\partial q_r} + \frac{\partial \Re}{\partial \dot{q}_r} = Q_r^*$$

→ $$\frac{\partial L}{\partial \dot{r}} = M\dot{r} \quad \frac{\partial L}{\partial \dot{\theta}} = Mr^2\dot{\theta}$$

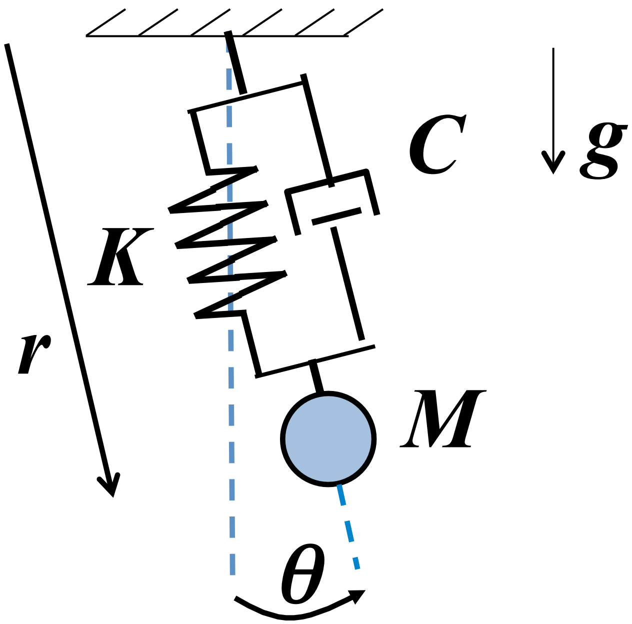
→ $$\frac{\partial L}{\partial r} = Mr\dot{\theta}^2 - K(r - r_u) + Mg\cos\theta \quad \frac{\partial L}{\partial \theta} = -Mgr\sin\theta$$

→ $$\frac{\partial \Re}{\partial \dot{r}} = C\dot{r} \quad \frac{\partial \Re}{\partial \dot{\theta}} = 0$$

→ $$r:\ M(\ddot{r} - r\dot{\theta}^2) + K(r - r_u) - Mg\cos\theta + C\dot{r} = 0$$

→ $$\theta:\ M(2r\dot{r}\dot{\theta} + r^2\ddot{\theta}) + Mgr\sin\theta = 0$$

Sprung Pendulum



$$\underbrace{M\left(\ddot{r} - r\dot{\theta}^2\right)}_{M\ddot{x}} + \underbrace{Kr}_{Kx} + \underbrace{C\dot{r}}_{C\dot{x}} = Kr_u + Mg\cos\theta$$

$$\underbrace{M\left(2r\dot{r}\dot{\theta} + r^2\ddot{\theta}\right)}_{M\ddot{x}} + \underbrace{Mgr\sin\theta}_{Kx} = 0$$

What if *r* is fixed?

$$Mr^2\ddot{\theta} + Mgr\sin\theta = 0$$

What if *θ* is fixed?

$$M\ddot{r} + C\dot{r} + Kr = Kr_u + Mg$$