ME 563
Mechanical Vibrations
Lecture #5

Lagrange's Method for
Deriving Equations of Motion

Derivation

Return to Newton's second law for a particle, *i*:

$$\longrightarrow \quad \mathbf{F}_i - M_i\ddot{\mathbf{r}}_i = \mathbf{0}$$

If we only consider the "active" forces, then we can "project" the equations onto the trajectory of the system to obtain the equation of motion as follows:

$$\longrightarrow \quad \sum_{i=1}^{N}\left(\mathbf{F}_{active,i} - M_i\ddot{\mathbf{r}}_i\right)\cdot \delta\mathbf{r}_i^{(k)} = 0$$

$\delta\mathbf{r}_i^{(k)}$ **is called the kinematic variation along the trajectory; we can express it in terms of displacement, velocity, etc.**

Derivation

When the variation is substituted into the previous equation,

$$\longrightarrow \sum_{i=1}^{N} \left(\mathbf{F}_{active,i} - M_i \ddot{\mathbf{r}}_i\right) \cdot \delta \mathbf{r}_i^{(k)} = 0 \longleftarrow \delta \mathbf{r}_i^{(k)} = \sum_{r=1}^{n} \frac{\partial \mathbf{r}_i^{(k)}}{\partial q_r^{(k)}} \delta q_r^{(k)}$$

Lagrange's equations of class II appear (after a lot of calculus).

This identity is needed:

$$\longrightarrow \ddot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_r} = \frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_r}\left(\frac{1}{2}\dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i\right)\right) - \frac{\partial}{\partial q_r}\left(\frac{1}{2}\dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i\right)$$

Proof is available upon request…

Derivation

When the forces are broken up into the active conservative and non-conservative forces, the formula for the so-called "non-conservative generalized forces" are found:

$$\rightarrow \sum_{i=1}^{N}\left(\mathbf{F}_{active,i}\right)\cdot\frac{\partial \mathbf{r}_i^{(k)}}{\partial q_r^{(k)}} = \sum_{i=1}^{N}\left(\mathbf{F}_{i,c}+\mathbf{F}_{i,nc}\right)\cdot\frac{\partial \mathbf{r}_i^{(k)}}{\partial q_r^{(k)}}$$

$$\rightarrow = \sum_{i=1}^{N}\left(-\nabla_i V+\mathbf{F}_{i,nc}\right)\cdot\frac{\partial \mathbf{r}_i^{(k)}}{\partial q_r^{(k)}}$$

Non-conservative generalized forces

$$Q_r^* = \sum_{i=1}^{N}\mathbf{F}_{i,nc}\cdot\frac{\partial \mathbf{r}_i^{(k)}}{\partial q_r^{(k)}}$$

$$\rightarrow = -\frac{\partial V}{\partial q_r}+Q_r^*$$

Derivation

Lastly, the formula for the kinematic motion is substituted into the equation of motion to obtain the final form:

$$\longrightarrow \quad \sum_{i=1}^{N} \left(M_i \ddot{\mathbf{r}}_i - \mathbf{F}_{active,i} \right) \cdot \delta \mathbf{r}_i^{(k)} = 0$$

$$\longrightarrow \quad \sum_{i=1}^{N} \left(M_i \ddot{\mathbf{r}}_i - \mathbf{F}_{active,i} \right) \cdot \sum_{r=1}^{n} \frac{\partial \mathbf{r}_i^{(k)}}{\partial q_r^{(k)}} \delta q_r^{(k)} = 0$$

$$\longrightarrow \quad \sum_{r=1}^{n} \sum_{i=1}^{N} \left(M_i \ddot{\mathbf{r}}_i - \mathbf{F}_{i,c} - \mathbf{F}_{i,nc} \right) \cdot \frac{\partial \mathbf{r}_i^{(k)}}{\partial q_r^{(k)}} \delta q_r^{(k)} = 0$$

$$\longrightarrow \quad \sum_{r=1}^{n} \left(\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_r} - \frac{\partial T}{\partial q_r} + \frac{\partial V}{\partial q_r} - Q_r^* \right) \delta q_r^{(k)} = 0$$

$$\longrightarrow \quad \sum_{r=1}^{n} \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_r} - \frac{\partial L}{\partial q_r} - Q_r^* \right) \delta q_r^{(k)} = 0$$

Lagrangian:

$$L = T - V$$

Derivation

If all of the generalized coordinates are independent, then

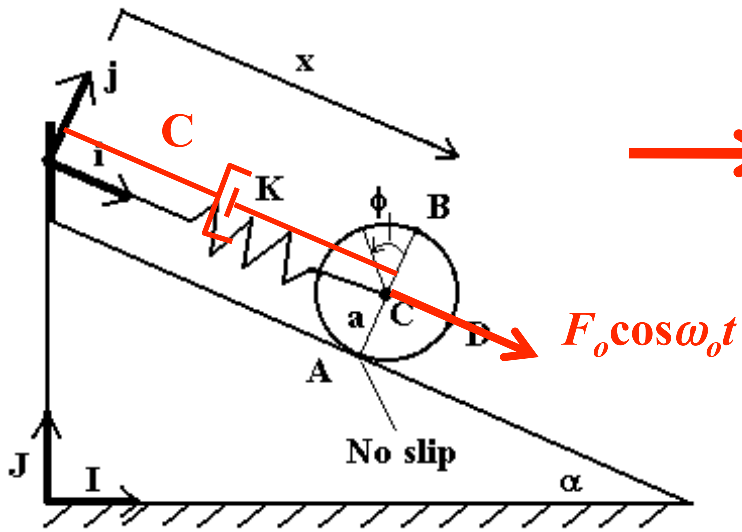
$$\rightarrow \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_r} - \frac{\partial L}{\partial q_r} = Q_r^*$$

for each of the r^{th} coordinates (degrees of freedom). An alternative form taken from the previous equations is,

Lagrange penalty

$$\rightarrow \quad \underbrace{\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_r} - \frac{\partial T}{\partial q_r}}_{M\ddot{x}} + \underbrace{\frac{\partial V}{\partial q_r}}_{Kx} = \underbrace{Q_r^*}_{-C\dot{x},\, f(t)}$$

Example
Rolling Disc on Incline



$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}I_{cm}\dot{\theta}^2$$

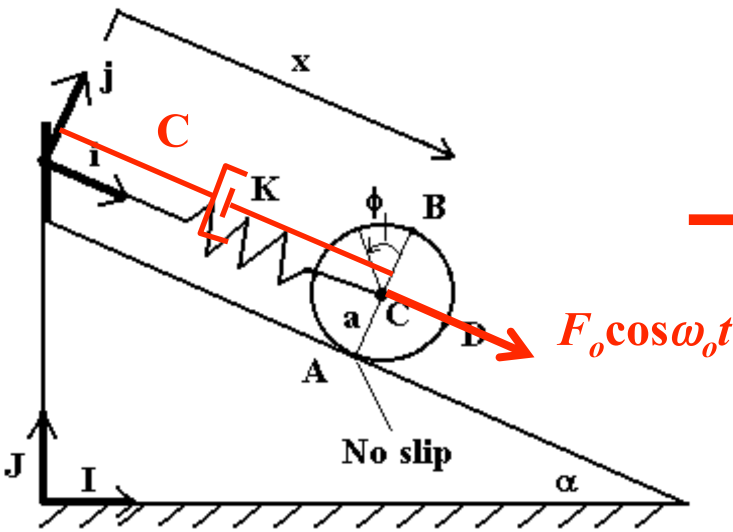
$$V = \frac{1}{2}K(x - x_u)^2 - Mg(x - x_u)\sin\alpha$$

Eliminate constraints

$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}I_{cm}\left(\frac{\dot{x}}{a}\right)^2$$

$$V = \frac{1}{2}K(x - x_u)^2 - Mg(x - x_u)\sin\alpha$$

Example
Rolling Disc on Incline



Calculate generalized forces:

$$\mathbf{r}_C = x\mathbf{i}$$

$$\frac{\partial \mathbf{r}_C}{\partial x} = \mathbf{i}$$

$$Q_r^* = \sum_{i=1}^{N} \left(\mathbf{F}_{i,nc}\right)\cdot \frac{\partial \mathbf{r}_i^{(k)}}{\partial q_r^{(k)}}$$

$$= \sum_{i=1}^{N} \left(F_o \cos\omega_o t - C\dot{x}\right)\mathbf{i} \cdot \frac{\partial \mathbf{r}_C}{\partial x}$$

$$= \left(F_o \cos\omega_o t - C\dot{x}\right)\mathbf{i} \cdot \mathbf{i}$$

$$= F_o \cos\omega_o t - C\dot{x}$$

Example
Rolling Disc on Incline

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_r} - \frac{\partial L}{\partial q_r} = Q_r^*$$

$$\frac{d}{dt}\frac{\partial}{\partial \dot{x}}\left(\frac{1}{2}M\dot{x}^2 + \frac{1}{2}I_{CM}\frac{\dot{x}}{a^2}^2 + Mg(x - x_u)\sin\alpha - \frac{1}{2}K(x - x_u)^2\right)$$

$$-\frac{\partial}{\partial x}\left(\frac{1}{2}M\dot{x}^2 + \frac{1}{2}I_{CM}\frac{\dot{x}}{a^2}^2 + Mg(x - x_u)\sin\alpha - \frac{1}{2}K(x - x_u)^2\right) = Q_r^*$$

$$\frac{d}{dt}\left(M\dot{x} + I_{CM}\frac{\dot{x}}{a^2}\right) - \left(Mg\sin\alpha - K(x - x_u)\right) = F_o\cos\omega_o t - C\dot{x}$$

$$\left(M + \frac{I_{CM}}{a^2}\right)\ddot{x} + C\dot{x} + K(x - x_u) - Mg\sin\alpha = F_o\cos\omega_o t$$