ME 563
Mechanical Vibrations
Lecture #22

Forced Response of Continuous Systems Using Modal Approach

String Forced Response

Consider the string equation of motion:

$$\rightarrow \quad \rho \frac{\partial^2 y(x,t)}{\partial t^2} - T \frac{\partial^2 y(x,t)}{\partial x^2} = f(x,t)$$

If we assume a solution of the form, $y(x,t) = Y(x) \cdot G(t)$ **, we can turn the partial differential equation into an ordinary differential equation of motion.**

To solve this ordinary differential equation, we can use the free response (eigenvalue solution) to decouple the individual modal equations from one another.

Recall Normal Coordinates

If we normalize the spatial functions, $Y_n(x)$, as follows:

$$\rightarrow \int_0^L \rho(x) Y_r(x) Y_s(x) dx = \delta_{rs} = \begin{cases} 0 & \text{for } r \neq s \\ 1 & \text{for } r = s \end{cases} \qquad Y_r(x) = \sqrt{\frac{2}{\rho L}} \sin\left(\frac{r\pi x}{L}\right)$$

$$\rightarrow \int_0^L T(x) \frac{dY_r(x)}{dx} \frac{dY_s(x)}{dx} dx = \omega_r^2 \delta_{rs} \qquad \omega_r = r\pi\sqrt{\frac{T}{\rho L^2}}$$

and we expand the solution as follows:

$$\rightarrow y(x,t) = \sum_{n=1}^{\infty} Y_n(x) G_n(t) = \sum_{n=1}^{\infty} Y_n(x) G_n \cos(\omega_n t + \phi_n)$$

then we multiply by one function $Y_r(x)$ at a time to uncouple the modes of vibration from one another.

→ $$\rho \frac{\partial^2 \sum_n Y_n(x) G_n(t)}{\partial t^2} - T \frac{\partial^2 \sum_n Y_n(x) G_n(t)}{\partial x^2} = f(x,t)$$

→ $$\rho \sum_n Y_n(x) \ddot{G}_n(t) - T \sum_n G_n(t) Y_n''(x) = f(x,t)$$

→ $$Y_r(x) \rho \sum_n Y_n(x) \ddot{G}_n(t) - T Y_r(x) \sum_n G_n(t) Y_n''(x) = Y_r(x) f(x,t)$$

→ $$\int_0^L Y_r(x) \rho \sum_n Y_n(x) \ddot{G}_n(t) dx - \int_0^L T Y_r(x) \sum_n G_n(t) Y_n''(x) dx = \int_0^L Y_r(x) f(x,t) dx$$

→ $$\ddot{G}_r(t) + \frac{\rho}{T} \int_0^L T Y_r(x) \sum_n \omega_n^2 G_n(t) Y_n(x) dx = \int_0^L Y_r(x) f(x,t) dx$$

→ $$\ddot{G}_r(t) + \omega_r^2 G_r(t) = \int_0^L Y_r(x) f(x,t) dx$$

Analogy to Discrete Case

Does this single degree of freedom equation remind you of anything?

→

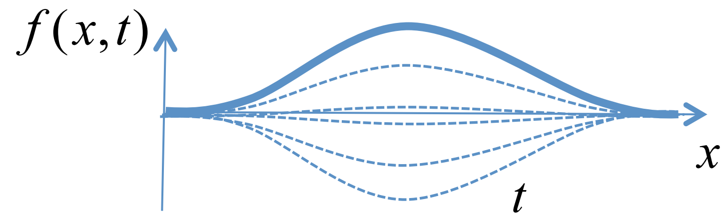
$$\ddot{G}_r(t) + \omega_r^2 G_r(t) = \int_0^L Y_r(x) f(x,t) dx \qquad r = 1,2,3,\ldots$$

→

$$[M_r]\{\ddot{p}\} + [C_r]\{\dot{p}\} + [K_r]\{p\} = [\Psi]^T \{f\}$$

<u>Both of these sets of equations describe how forcing functions (in space and time) excite uncoupled modes of vibration.</u>

Example *f(x,t)*



If we assume a forcing function,

$$f(x,t) = \left[1 - \cos\left(\frac{2x\pi}{L}\right)\right]\sin\omega t$$

then we can arrive at the following set of equations:

$$\ddot{G}_r(t) + \omega_r^2 G_r(t) = \int_0^L \sqrt{\frac{2}{\rho L}} \sin\left(\frac{r\pi}{L}x\right) f(x,t)\,dx \qquad r = 1,2,3,\ldots$$

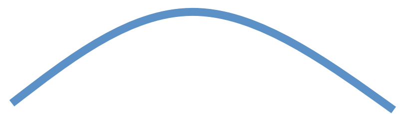
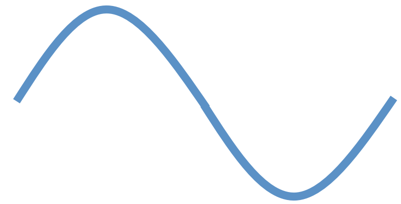
$$\ddot{G}_r(t) + \omega_r^2 G_r(t) = \sqrt{\frac{2}{\rho L}} \int_0^L \sin\left(\frac{r\pi}{L}x\right)\left[1 - \cos\left(\frac{2\pi}{L}x\right)\right]\sin(\omega t)\,dx$$

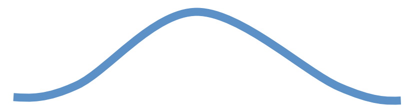
$$\ddot{G}_r(t) + \omega_r^2 G_r(t) = \sqrt{\frac{2}{\rho L}} \sin(\omega t)\frac{L}{r\pi}\left[1 - \cos(r\pi)\right]$$

Example

The meaning of this solution,

$$\ddot{G}_r(t) + \omega_r^2 G_r(t) = \sqrt{\frac{2}{\rho L}}\sin(\omega t)\frac{L}{r\pi}\left[1 - \cos(r\pi)\right]$$

is that only modes like  **not** 

are excited by a spatial forcing function like  **.**

Also, modes of vibration with natural frequencies, ω_r , close to the forcing frequency, ω , are excited the most.

Numerator

Note the analogy to FRFs… $$\ddot{G}_r(t) + \omega_r^2 G_r(t) = \sqrt{\frac{2}{\rho L}}\sin(\omega t)\frac{L}{r\pi}\left[1 - \cos(r\pi)\right]$$

Denominator