ME 563
Mechanical Vibrations
Lecture #21

Multiple Degree of Freedom

Modal Approach for Forced Response

Modal Approach

We have considered one method for obtaining the forced response of a multiple degree of freedom system based on the use of frequency response functions.

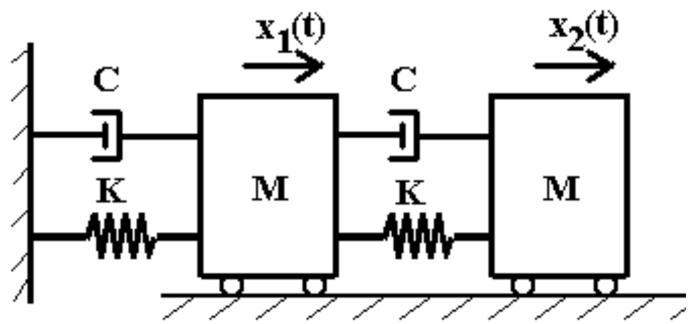
$$\rightarrow \begin{Bmatrix} X_1(j\omega) \\ X_2(j\omega) \end{Bmatrix} = \begin{bmatrix} H_{11}(j\omega) & H_{12}(j\omega) \\ H_{21}(j\omega) & H_{22}(j\omega) \end{bmatrix} \begin{Bmatrix} F_1(j\omega) \\ F_2(j\omega) \end{Bmatrix}$$

This approach produces the steady state response of the physical coordinates, but what if we are interested in the modal responses instead?

The modal approach to forced response can be used to decouple the equations of motion and solve for the individual modal responses.

Transforming Coordinates

Consider the system shown below.



The equations of motion are given by:

$$\begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 2C & -C \\ -C & C \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} 2K & -K \\ -K & K \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} f_1(t) \\ f_2(t) \end{Bmatrix}$$

and we uncouple these equations using the substitution,

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{bmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{bmatrix} \begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix}$$

Uncoupling EOMs

The equations of motion are uncoupled as follows,

→ $$[\Psi]^T[M][\Psi]\{\ddot{p}\} + [\Psi]^T[C][\Psi]\{\dot{p}\} + [\Psi]^T[K][\Psi]\{p\} = [\Psi]^T\{f\}$$

leading to the following forced modal equations:

→ $$M_{r1}\ddot{p}_1 + C_{r1}\dot{p}_1 + K_{r1}p_1 = f_{r1}(t)$$
$$M_{r2}\ddot{p}_2 + C_{r2}\dot{p}_2 + K_{r2}p_2 = f_{r2}(t)$$

The modal forces are given by,

→ $$f_{r1}(t) = \psi_{11}f_1 + \psi_{21}f_2$$
$$f_{r2}(t) = \psi_{12}f_1 + \psi_{22}f_2$$

What is the Meaning?

$$\rightarrow \quad M_{r1}\ddot{p}_1 + C_{r1}\dot{p}_1 + K_{r1}p_1 = \psi_{11}f_1(t) + \psi_{21}f_2(t)$$
$$M_{r2}\ddot{p}_2 + C_{r2}\dot{p}_2 + K_{r2}p_2 = \psi_{12}f_1(t) + \psi_{22}f_2(t)$$

These equations indicate that:

- The forced response of each mode is independent of the other modes.

- The forced response of each mode is determined by
(a) a combination of the applied forces,
(b) the system parameters associated with that mode.

- The modal vector coefficients serve as a filter between the physically applied forces and the modes of vibration.

Solving for Responses

$$M_{r1}\ddot{p}_1 + C_{r1}\dot{p}_1 + K_{r1}p_1 = \psi_{11}f_1(t) + \psi_{21}f_2(t)$$
$$M_{r2}\ddot{p}_2 + C_{r2}\dot{p}_2 + K_{r2}p_2 = \psi_{12}f_1(t) + \psi_{22}f_2(t)$$

To solve for the response to,

$$f_1(t) = F_1 \sin\omega t,\ f_2(t) = 0$$

the individual equations are solved as follows:

$$p_1(t) = \psi_{11}F_1\left|H_{r1r1}\right|_{\omega}\sin\left(\omega t + \left(\angle H_{r1r1}\right)_{\omega}\right)$$
$$p_2(t) = \psi_{12}F_1\left|H_{r2r2}\right|_{\omega}\sin\left(\omega t + \left(\angle H_{r2r2}\right)_{\omega}\right)$$

where,

$$H_{r1r1}(\omega) = \frac{1}{K_{r1} - M_{r1}\omega^2 + j\omega C_{r1}} \quad H_{r2r2}(\omega) = \frac{1}{K_{r2} - M_{r2}\omega^2 + j\omega C_{r2}}$$

Combinations of Forces

$$M_{r1}\ddot{p}_1 + C_{r1}\dot{p}_1 + K_{r1}p_1 = \psi_{11}f_1(t) + \psi_{21}f_2(t)$$
$$M_{r2}\ddot{p}_2 + C_{r2}\dot{p}_2 + K_{r2}p_2 = \psi_{12}f_1(t) + \psi_{22}f_2(t)$$

What if the forces are given by?

$$f_1(t) = \psi_{21}\sin\omega t,\ f_2(t) = -\psi_{11}\sin\omega t$$

The response at mode #1 is zero because the modal force for that mode is zero,

$$\psi_{11}f_1(t) + \psi_{21}f_2(t) = 0$$

This result suggests that a system can be designed in such as way that certain modes are eliminated from the response.

Relating the Two Methods

→ $$H_{r1r1}(\omega) = \frac{1}{K_{r1} - M_{r1}\omega^2 + j\omega C_{r1}} \qquad H_{r2r2}(\omega) = \frac{1}{K_{r2} - M_{r2}\omega^2 + j\omega C_{r2}}$$

These frequency response functions must be related to the frequency response functions we used before:

→ $$\begin{Bmatrix} X_1(j\omega) \\ X_2(j\omega) \end{Bmatrix} = \begin{bmatrix} H_{11}(j\omega) & H_{12}(j\omega) \\ H_{21}(j\omega) & H_{22}(j\omega) \end{bmatrix} \begin{Bmatrix} F_1(j\omega) \\ F_2(j\omega) \end{Bmatrix}$$

To relate the two sets of functions, we write down the physical coordinates in terms of those functions and equate the results.

Relating the Two Methods

$$X_1 = H_{11}F_1 + H_{12}F_2$$
$$X_2 = H_{21}F_1 + H_{22}F_2$$

$$X_1 = \psi_{11}P_1 + \psi_{12}P_2$$
$$= \psi_{11}H_{r1r1}\psi_{11}F_1 + \psi_{11}H_{r1r1}\psi_{21}F_2$$
$$+ \psi_{12}H_{r2r2}\psi_{12}F_1 + \psi_{12}H_{r2r2}\psi_{22}F_2$$

$$X_2 = \psi_{21}P_1 + \psi_{22}P_2$$
$$= \psi_{21}H_{r1r1}\psi_{11}F_1 + \psi_{21}H_{r1r1}\psi_{21}F_2$$
$$+ \psi_{22}H_{r2r2}\psi_{12}F_1 + \psi_{22}H_{r2r2}\psi_{22}F_2$$

Based on these expressions, we see that the two sets of frequency response functions are related through a modal decomposition (sum of the modes of vibration).

Modal Decomposition

→ $$X_1 = H_{11}F_1 + H_{12}F_2$$

→ $$\begin{aligned} X_1 &= \psi_{11}P_1 + \psi_{12}P_2 \\ &= \psi_{11}H_{r1r1}\psi_{11}F_1 + \psi_{11}H_{r1r1}\psi_{21}F_2 \\ &\quad + \psi_{12}H_{r2r2}\psi_{12}F_1 + \psi_{12}H_{r2r2}\psi_{22}F_2 \end{aligned}$$

The physical frequency response functions can be decomposed in terms of the modal FRFs:

→ $$\begin{aligned} X_1 &= (\psi_{11}H_{r1r1}\psi_{11} + \psi_{12}H_{r2r2}\psi_{12})F_1 \\ &\quad + (\psi_{11}H_{r1r1}\psi_{21} + \psi_{12}H_{r2r2}\psi_{22})F_2 \end{aligned}$$

→ $$\begin{aligned} H_{11} &= \psi_{11}H_{r1r1}\psi_{11} + \psi_{12}H_{r2r2}\psi_{12} \\ H_{12} &= \psi_{11}H_{r1r1}\psi_{21} + \psi_{12}H_{r2r2}\psi_{22} \end{aligned}$$

Modal Superposition

This result shows that the frequency response functions are sums of modal frequency response functions:

$$\rightarrow \quad H_{11} = \sum_{m=1}^{2} \psi_{1m} H_{rmrm} \psi_{1m}$$

$$\rightarrow \quad H_{12} = \sum_{m=1}^{2} \psi_{1m} H_{rmrm} \psi_{2m} = H_{21}$$

$$\rightarrow \quad H_{22} = \sum_{m=1}^{2} \psi_{2m} H_{rmrm} \psi_{2m}$$