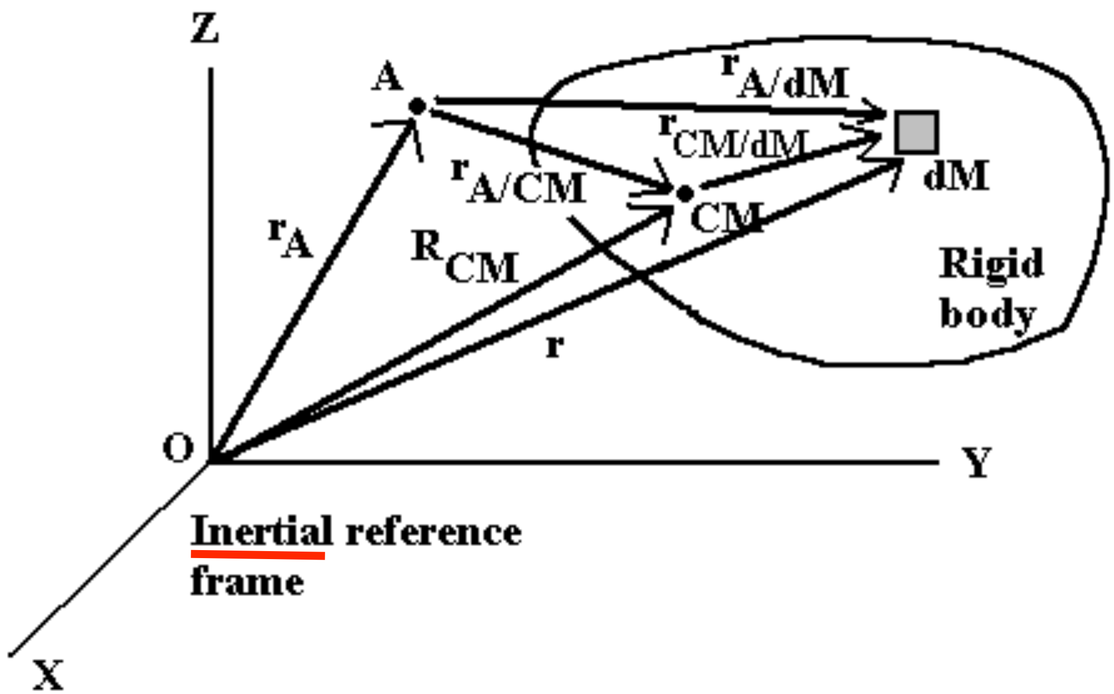
ME 563
Mechanical Vibrations
Lecture #2

Newton-Euler Laws

(Derivation of equations of motion)

Newton-Euler Laws



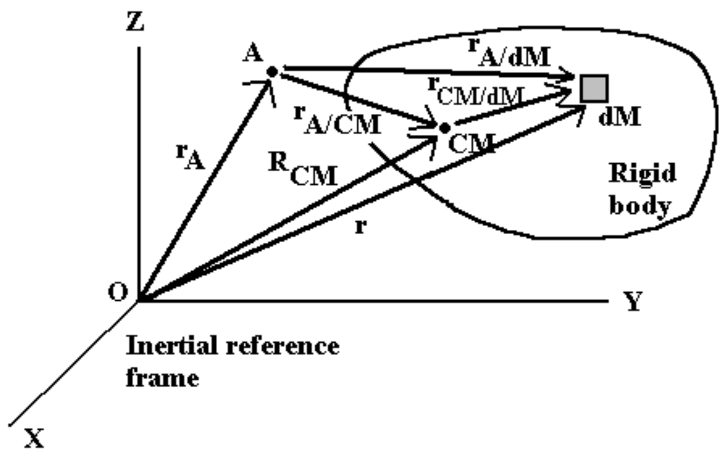
$$\mathbf{F}_{CM} = \frac{d\mathbf{P}_{CM}}{dt} = M\ddot{\mathbf{R}}_{CM}$$

where $\mathbf{P}_{CM} = M\mathbf{V}_{CM}$

$$\mathbf{M}_{CM} = \frac{d\mathbf{H}_{CM}}{dt}$$

where $\mathbf{H}_{CM} = \mathbf{R}_{CM} \times \mathbf{P}_{CM}$

Flavors of Euler's Law



$$\mathbf{M}_A = \frac{d\mathbf{H}_A}{dt} + \dot{\mathbf{R}}_A \times M\dot{\mathbf{R}}_{CM}$$

$$= \frac{d\mathbf{H}_{CM}}{dt} + \mathbf{r}_{A/CM} \times M\ddot{\mathbf{R}}_{CM}$$

$$= \frac{d\mathbf{H}_A}{dt} + \mathbf{r}_{A/CM} \times M\ddot{\mathbf{R}}_A$$

What point "A" should we use?

A = CM
A is fixed
A moves parallel to CM

Proof of the Previous Slide (first two lines)

Parallel axis theorem ($\mathbf{I_A = I_{cm} + M \times r_{A/cm}^2}$)

$$\mathbf{M}_A = \frac{d\mathbf{H}_A}{dt} + \dot{\mathbf{R}}_A \times M\dot{\mathbf{R}}_{CM}$$

$$= \frac{d}{dt}\left(\mathbf{H}_{CM} + \mathbf{r}_{A/CM} \times M\dot{\mathbf{R}}_{CM} \right) + \dot{\mathbf{R}}_A \times M\dot{\mathbf{R}}_{CM}$$

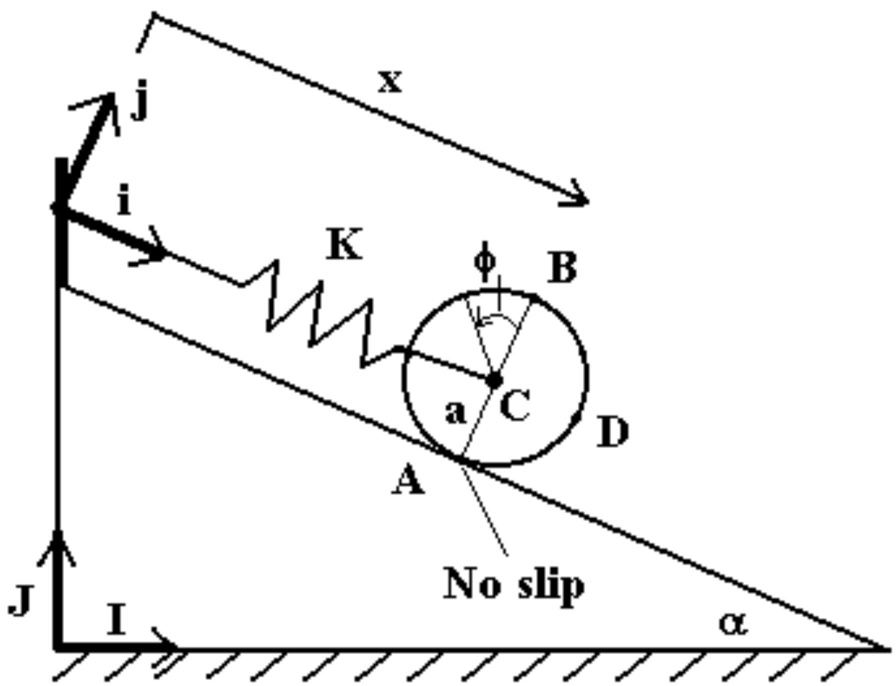
$$= \frac{d\mathbf{H}_{CM}}{dt} + \dot{\mathbf{r}}_{A/CM} \times M\dot{\mathbf{R}}_{CM} + \mathbf{r}_{A/CM} \times M\ddot{\mathbf{R}}_{CM} + \dot{\mathbf{R}}_A \times M\dot{\mathbf{R}}_{CM}$$

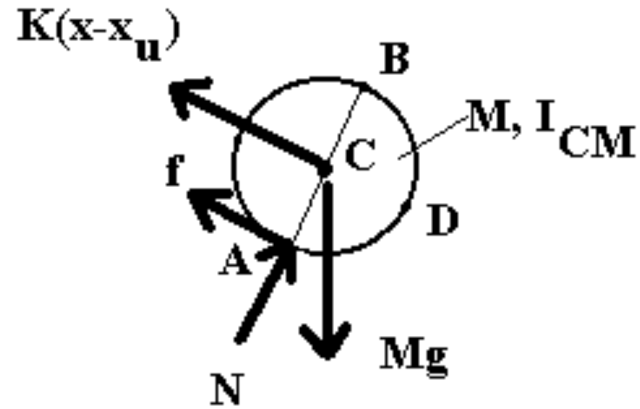
$$= \frac{d\mathbf{H}_{CM}}{dt} + \dot{\mathbf{r}}_{A/CM} \times M\dot{\mathbf{R}}_{CM} + \dot{\mathbf{R}}_A \times M\dot{\mathbf{R}}_{CM} + \mathbf{r}_{A/CM} \times M\ddot{\mathbf{R}}_{CM}$$

$$= \frac{d\mathbf{H}_{CM}}{dt} + \left(\dot{\mathbf{R}}_{CM} - \dot{\mathbf{R}}_A\right) \times M\dot{\mathbf{R}}_{CM} + \dot{\mathbf{R}}_A \times M\dot{\mathbf{R}}_{CM} + \mathbf{r}_{A/CM} \times M\ddot{\mathbf{R}}_{CM}$$

$$= \frac{d\mathbf{H}_{CM}}{dt} + \mathbf{r}_{A/CM} \times M\ddot{\mathbf{R}}_{CM}$$

Example
Rolling Disc on Incline





K is massless and linear
No slip (velocity of point in disc at incline is zero)

Example
Rolling Disc on Incline

$$\mathbf{M}_{CM} = \frac{d\mathbf{H}_{CM}}{dt}$$

$$(-a\mathbf{j})\times(-f\mathbf{i}) = \frac{d}{dt}\left(I_{CM}\dot{\phi}\mathbf{k}\right)$$

$$-af\mathbf{k} = I_{CM}\ddot{\phi}\mathbf{k} \Rightarrow I_{CM}\ddot{\phi} = -af$$

Euler's Law about Center of Mass

Newton's Law

$$\mathbf{F}_{CM} = M\mathbf{A}_{CM}$$

$$(\cdot)\mathbf{j} + \left(-K(x - x_u) - f + Mg\sin\alpha\right)\mathbf{i} = M(\ddot{x})\mathbf{i}$$

$$-M\ddot{x} + Mg\sin\alpha - K(x - x_u) = f$$

Applying Constraints

$x=-a\cdot\phi$ (rolling constraint)

$$I_{CM}\ddot{\phi} = -af$$

$$Ma\ddot{\phi} + Mg\sin\alpha + K\left(a\phi + x_u\right) = f$$

Two equations, two unknowns

$$Ma\ddot{\phi} + Mg\sin\alpha + K\left(a\phi + x_u\right) = -\frac{1}{a}I_{cm}\ddot{\phi}$$

$$\left(I_{cm} + Ma^2\right)\ddot{\phi} + Mga\sin\alpha + Ka\left(a\phi + x_u\right) = 0$$

$$\left(I_{cm} + Ma^2\right)\ddot{\phi} + +Ka^2\phi = -Mga\sin\alpha - Kax_u$$

Example
Rolling Disc on Incline

Euler's Law about Point A

$$\mathbf{M}_A = \frac{d\mathbf{H}_A}{dt} + \dot{\mathbf{r}}_A \times M\dot{\mathbf{R}}_{CM} = \frac{d\mathbf{H}_A}{dt}$$

$$(a\mathbf{j}) \times (-Mg\mathbf{J}) + (a\mathbf{j}) \times \left(-K(x - x_u)\mathbf{i}\right) = \frac{d}{dt}\left(I_A \dot{\phi}\mathbf{k}\right)$$

$$-aMg\sin\alpha\mathbf{k} + aK(x - x_u)\mathbf{k} = I_{CM}\ddot{\phi}\mathbf{k} \Rightarrow I_A\ddot{\phi} = -aMg\sin\alpha + aK(x - x_u)$$

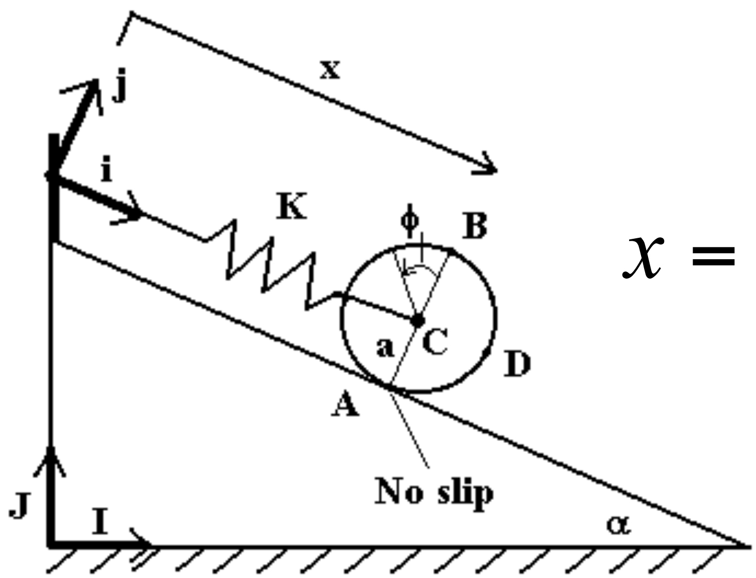
Euler's Law about Point B

$$\mathbf{M}_B = \frac{d\mathbf{H}_B}{dt} + \dot{\mathbf{r}}_B \times M\dot{\mathbf{R}}_{CM} = \frac{d\mathbf{H}_B}{dt}$$

$$(-2a\mathbf{j}) \times (-f)\mathbf{i} + (-a\mathbf{j}) \times \left(-Mg\mathbf{J} - K(x - x_u)\mathbf{i}\right) = \frac{d}{dt}\left(I_B \dot{\phi}\mathbf{k}\right)$$

$$-2af\mathbf{k} + aMg\sin\alpha\mathbf{k} - aK(x - x_u)\mathbf{k} = I_{CM}\ddot{\phi}\mathbf{k} \Rightarrow I_B\ddot{\phi} = -2af + aMg\sin\alpha - aK(x - x_u)$$

Example
Rolling Disc on Incline



$$x = -a\phi$$

What if we choose *x* as our independent variable?

$$\left(I_{CM} + Ma^2\right)\ddot{\phi} + Ka^2\phi = -Mga\sin\alpha - aKx_u$$

How do we check the equation of motion?

Example
Rolling Disc on Incline

To eliminate static terms from the equation of motion, we can transform our coordinates:

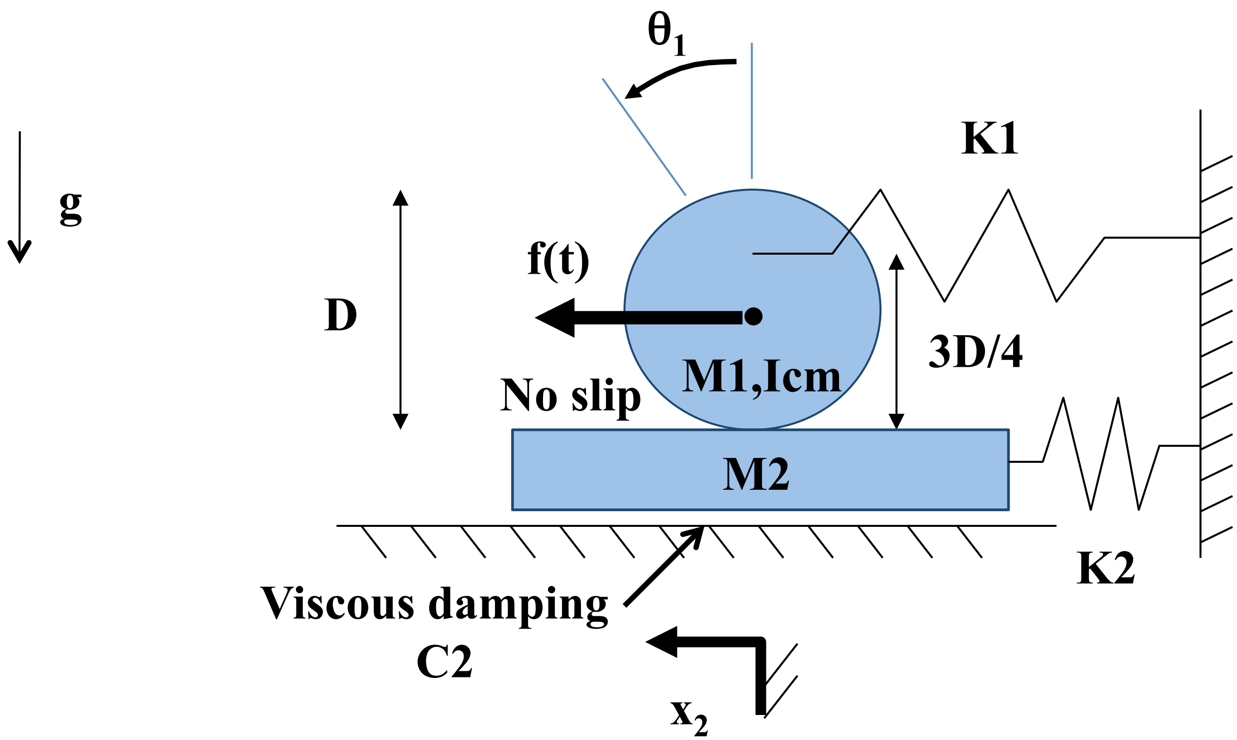
→ $$\phi = \phi_{dynamic} + \phi_{static}$$

→ $$\left(I_{CM} + Ma^2\right)\ddot{\phi}_{dynamic} + Ka^2\left(\phi_{dynamic} + \phi_{static}\right) = -Mga\sin\alpha - aKx_u$$

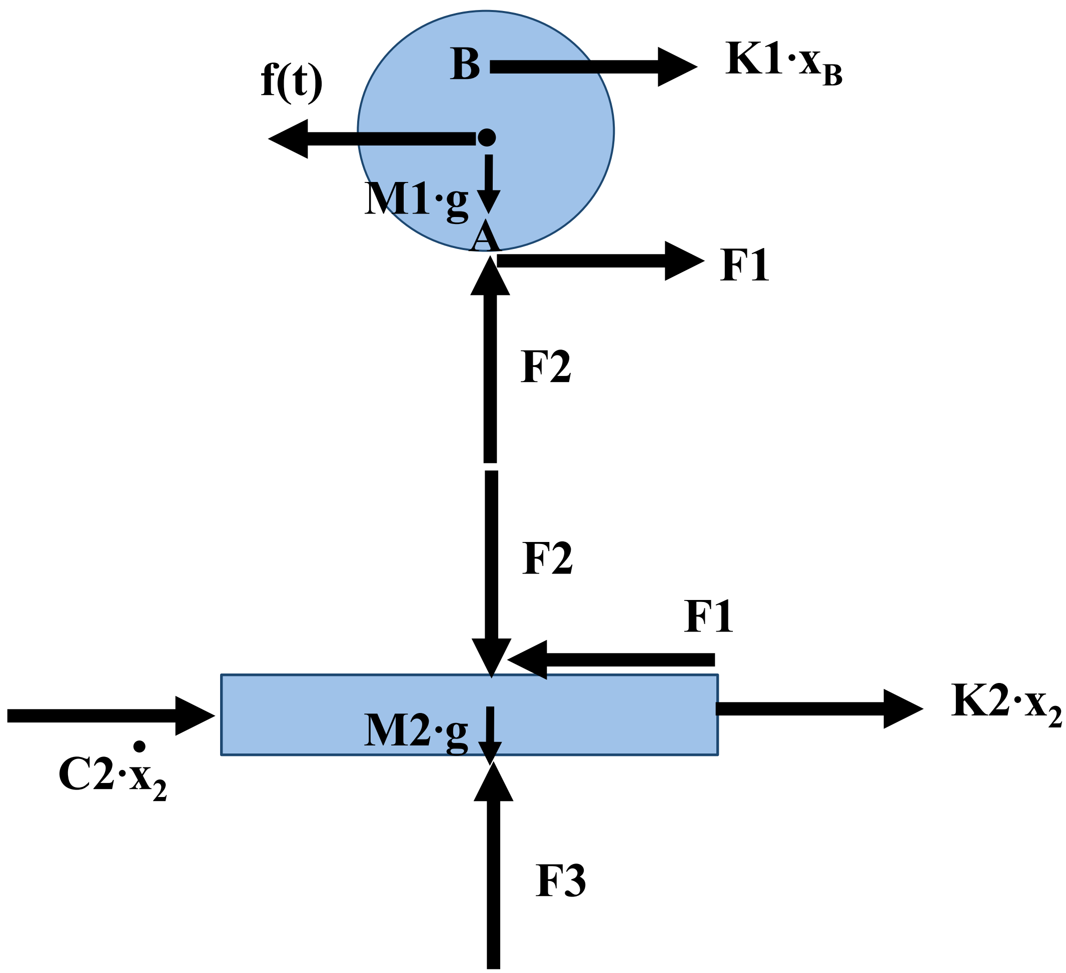
→ $$\left(I_{CM} + Ma^2\right)\ddot{\phi}_{dynamic} + Ka^2\phi_{dynamic} = \underbrace{-Mga\sin\alpha - aKx_u - Ka^2\phi_{static}}_{\textit{Static}\text{ free body diagram}}$$

→ $$\left(I_{CM} + Ma^2\right)\ddot{\phi}_{dynamic} + Ka^2\phi_{dynamic} = 0$$

Example
Rolling Disc on a Moving Cart



Example
Rolling Disc on a Moving Cart



Example
Rolling Disc on a Moving Cart

$$\underset{\curvearrowleft}{+}\sum_{I_{cm}} T = I_{cm}\ddot{\theta}_1 = -\frac{D}{4}K_1\left(x_1 + x_{B/cm}\right) + \frac{D}{2}F_1$$

$$\underset{\leftarrow}{+}\sum_{M2} F = M_2\ddot{x}_2 = -C_2\dot{x}_2 - K_2x_2 + F_1$$

$$\underset{\leftarrow}{+}\sum_{M1} F = M_1\ddot{x}_1 = -K_1\left(x_1 + x_{B/cm}\right) - F_1 + f(t)$$

$$x_{B/cm} = \frac{D}{4}\theta_1,\;\; x_1 = x_2 + \frac{D}{2}\theta_1$$

5 equations
5 unknowns

Example
Rolling Disc on a Moving Cart

$$\left(I_{cm} + M_1\frac{D^2}{4}\right)\ddot{\theta}_1 + \frac{9D^2}{16}K_1\theta_1 + \frac{D}{2}M_1\ddot{x}_2 + \frac{3D}{4}K_1x_2 = \frac{D}{2}f(t)$$

$$(M_1 + M_2)\ddot{x}_2 + C_2\dot{x}_2 + (K_1 + K_2)x_2 + \frac{D}{2}M_1\ddot{\theta}_1 + K_1\frac{3D}{4}\theta_1 = f(t)$$

$$\begin{bmatrix} I_{cm} + M_1\frac{D^2}{4} & \frac{D}{2}M_1 \\ \frac{D}{2}M_1 & M_1 + M_2 \end{bmatrix}\begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & C_2 \end{bmatrix}\begin{Bmatrix} \dot{\theta}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} \frac{9D^2}{16}K_1 & \frac{3D}{4}K_1 \\ \frac{3D}{4}K_1 & K_1 + K_2 \end{bmatrix}\begin{Bmatrix} \theta_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} D/2 \\ 1 \end{Bmatrix}f(t)$$