ME 563
Mechanical Vibrations
Lecture #18

Multiple Degree of Freedom

Frequency Response Functions

Frequency Response

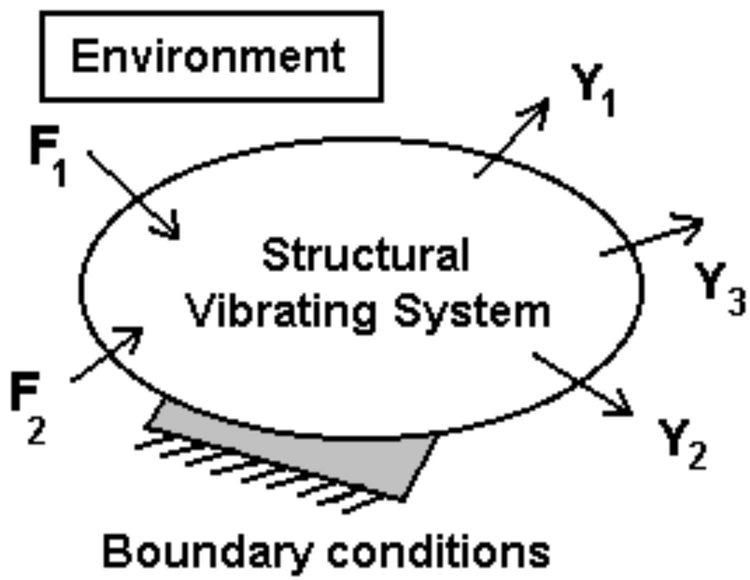
When we considered a single degree of freedom system with one input force and one output response, the relationship between the steady state response and force was written as:

$$\rightarrow \quad X_p(\omega) = H(\omega)F(\omega)$$

This system is called a single input single output (SISO) system.

This relationship can also be developed for multiple degree of freedom systems with more than one input and output. In this type of multiple input multiple output (MIMO) system, there is more than one frequency response function.

Frequency Response



Total response = **Sum of responses to individual forces at each frequency**

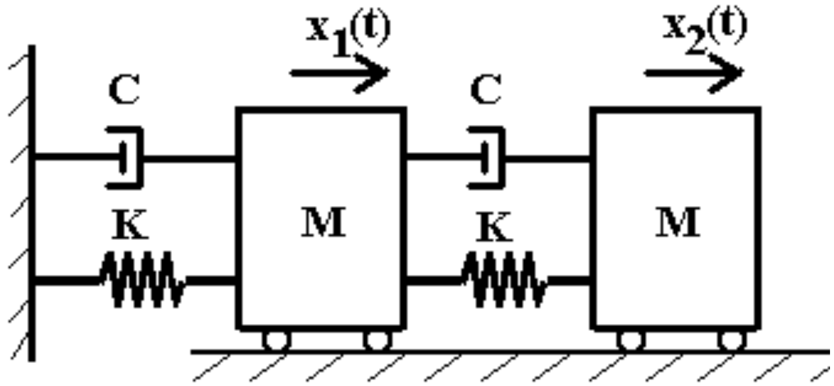
$$Y_1 = H11 * F_1 + H12 * F_2$$

$$Y_2 = H21 * F_1 + H22 * F_2$$

$$Y_3 = H31 * F_1 + H32 * F_2$$

MDOF System

Consider the system shown below.



The equations of motion are given by:

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$$\begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 2C & -C \\ -C & C \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} 2K & -K \\ -K & K \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix}$$

Transfer Functions

After applying the Laplace transform, to these equations of motion, the following transfer function relationships are found:

$$\begin{Bmatrix} X_1(s) \\ X_2(s) \end{Bmatrix} = \left[H(s)\right]\begin{Bmatrix} F_1(s) \\ F_2(s) \end{Bmatrix}$$

$$= \frac{1}{\Delta(s)}\begin{bmatrix} Ms^2 + Cs + K & Cs + K \\ Cs + K & Ms^2 + 2Cs + 2K \end{bmatrix}\begin{Bmatrix} F_1(s) \\ F_2(s) \end{Bmatrix}$$

where $\Delta(s) = \left(Ms^2 + 2Cs + 2K\right)\left(Ms^2 + Cs + K\right) - \left(Cs + K\right)^2$

Note that the denominator Δ(s) is the same for every transfer function; i.e., natural frequencies are properties of the system.

Transfer Functions

The transfer functions are written using subscripts, where the first subscript denotes the response degree of freedom and the second subscript denotes the input degree of freedom:

$$\rightarrow \begin{Bmatrix} X_1(s) \\ X_2(s) \end{Bmatrix} = \begin{bmatrix} H_{11}(s) & H_{12}(s) \\ H_{21}(s) & H_{22}(s) \end{bmatrix} \begin{Bmatrix} F_1(s) \\ F_2(s) \end{Bmatrix}$$

To obtain the frequency response functions, we substitute $s=j\omega$ as we did previously for single degree of freedom case:

Frequency Response Functions

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$$\begin{Bmatrix} X_1(j\omega) \\ X_2(j\omega) \end{Bmatrix} = \begin{bmatrix} H_{11}(j\omega) & H_{12}(j\omega) \\ H_{21}(j\omega) & H_{22}(j\omega) \end{bmatrix} \begin{Bmatrix} F_1(j\omega) \\ F_2(j\omega) \end{Bmatrix}$$

where $H_{11}(j\omega) = \left(K - M\omega^2 + j\omega C\right)/\Delta(j\omega)$

$$H_{12}(j\omega) = H_{21}(j\omega) = \left(K + j\omega C\right)/\Delta(j\omega)$$

$$H_{22}(j\omega) = \left(2K - M\omega^2 + j\omega 2C\right)/\Delta(j\omega)$$

These frequency response functions can be utilized to calculate the sinusoidal responses of a multiple degree of freedom system as we did in the single degree of freedom case.

Using the FRFs

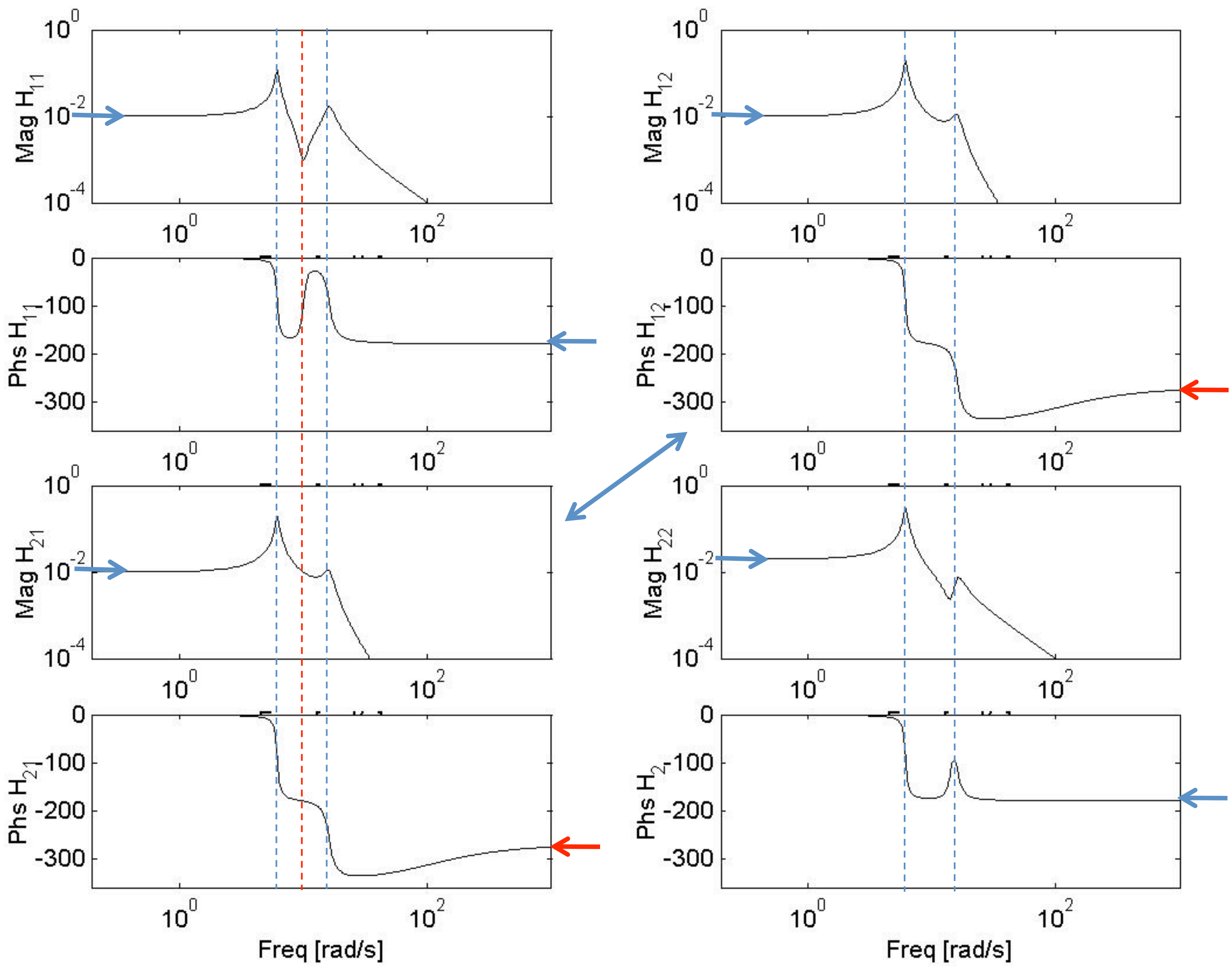
If $f_1(t) = F_{1i}\cos(\omega_1 t + \phi_{i1})$ and $f_2(t) = F_{2i}\cos(\omega_2 t + \phi_{i2})$

then,

$$x_{1p}(t) = \|H_{11}(j\omega_1)\|F_{1i}\cos(\omega_1 t + \phi_{i1} + \angle H_{11}(j\omega_1))$$
$$+ \|H_{12}(j\omega_2)\|F_{2i}\cos(\omega_2 t + \phi_{i2} + \angle H_{12}(j\omega_2))$$

$$x_{2p}(t) = \|H_{21}(j\omega_1)\|F_{1i}\cos(\omega_1 t + \phi_{i1} + \angle H_{21}(j\omega_1))$$
$$+ \|H_{22}(j\omega_2)\|F_{2i}\cos(\omega_2 t + \phi_{i2} + \angle H_{22}(j\omega_2))$$

Plots of FRFs



Notes about Plots of FRFs

Low frequency response corresponds to static response.

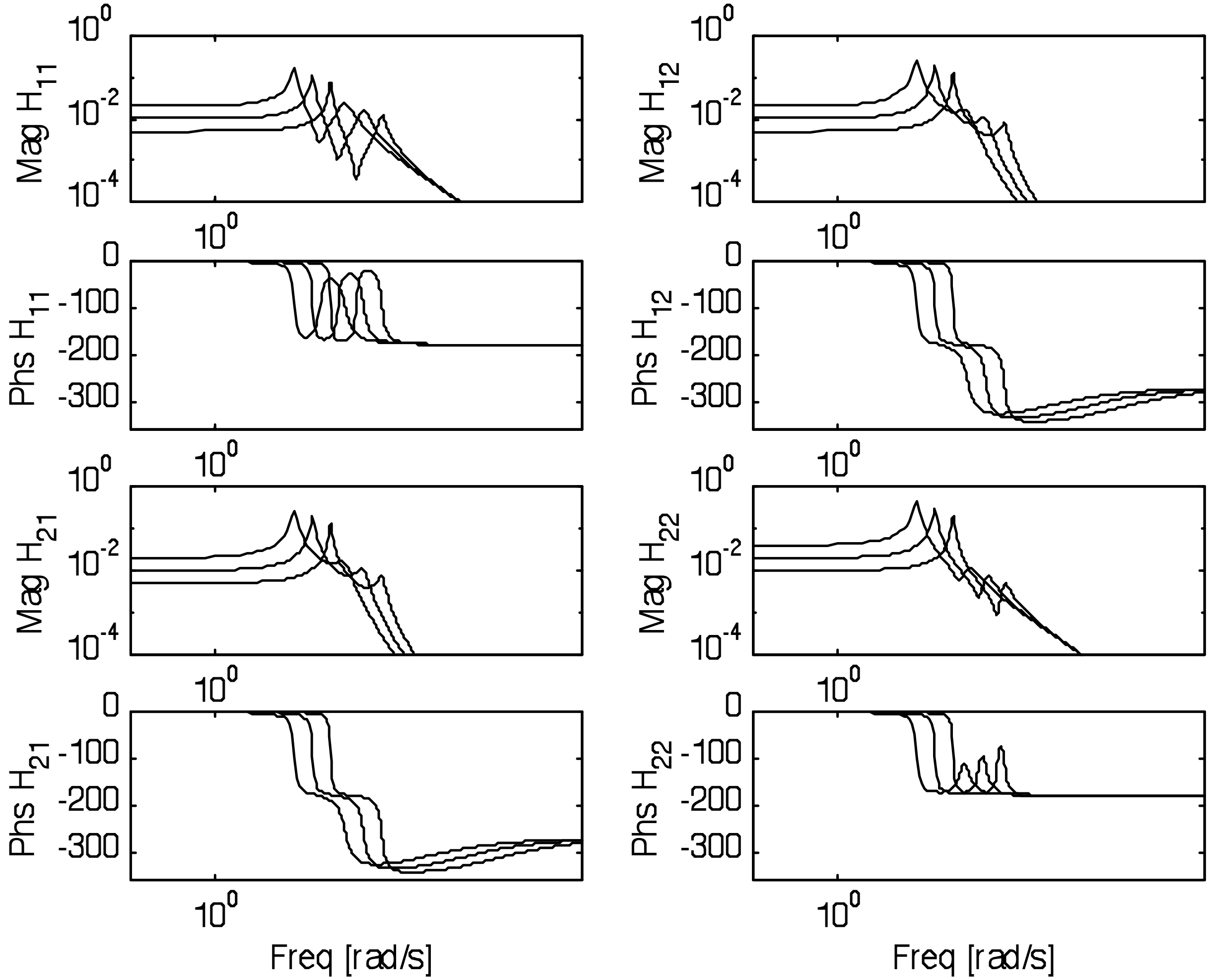
Resonant frequencies are the same for all FRFs.

Anti-resonant frequencies (zeros) only occur in driving point FRFs (for which input and output are the same).

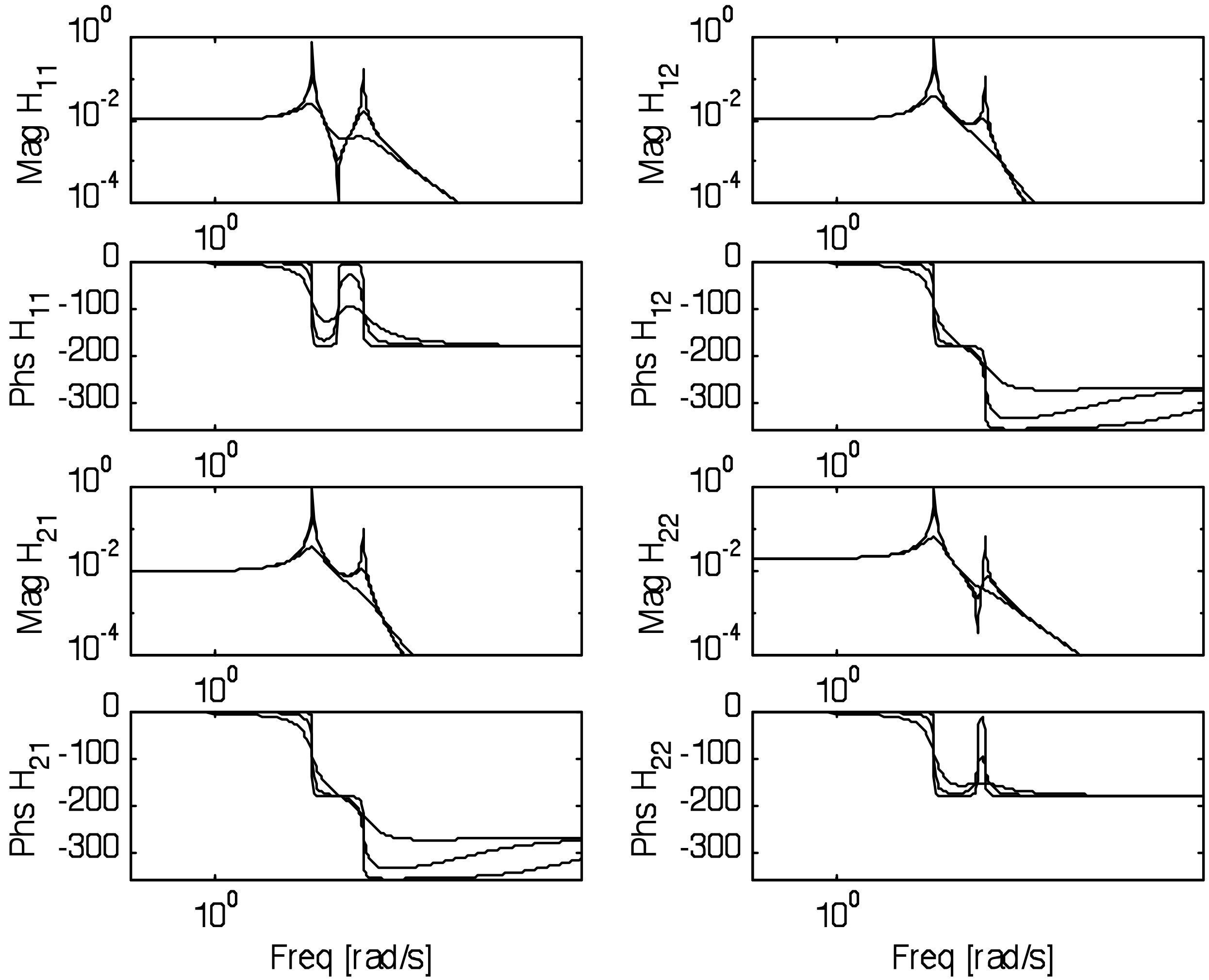
Phase recovers after a resonance in driving point FRF but is lost through both resonant frequencies in cross point FRFs.

Cross point FRFs are equal so long as the system is conservative and absolute coordinates are utilized.

Sensitivity of FRFs



Sensitivity of FRFs



Sensitivity of FRFs