ME 563
Mechanical Vibrations
Lecture #17

Frequency Response Functions

Sinusoidal Response

When we considered a co-sinusoidal input force, $F_i \cos(\omega t + \phi_i)$ **we obtained the following amplitude/phase information:**

→ $x_p(t) = X_p \cos(\omega t + \varphi_p)$

→ $$X_p = \frac{F_i}{\sqrt{\left[K - \left(M + \frac{I_{CM}}{a^2}\right)\omega^2\right]^2 + (\omega C)^2}}$$

→ $$\phi_p = \phi_i - \tan^{-1} \frac{\omega C}{K - \left(M + \frac{I_{CM}}{a^2}\right)\omega^2}$$

We can also use Laplace transforms to find these amplitude and phase relationships.

Transfer Functions

When we considered a co-sinusoidal input force, $F_i \cos(\omega t + \phi_i)$ **we obtained the following amplitude/phase information:**

→ $$\left[\left(M + \frac{I_{CM}}{a^2}\right)s^2 + Cs + K\right]X_d(s) = \left(M + \frac{I_{CM}}{a^2}\right)s x_d(0) + C x_d(0) + \left(M + \frac{I_{CM}}{a^2}\right)\dot{x}_d(0) + F(s)$$

→ $$X_d(s) = \frac{\left(M + \frac{I_{CM}}{a^2}\right)s + C}{\left(M + \frac{I_{CM}}{a^2}\right)s^2 + Cs + K} x_d(0) + \frac{\left(M + \frac{I_{CM}}{a^2}\right)}{\left(M + \frac{I_{CM}}{a^2}\right)s^2 + Cs + K}\dot{x}_d(0) + \frac{1}{\left(M + \frac{I_{CM}}{a^2}\right)s^2 + Cs + K}F(s)$$

→ Impedance function: $B(s) = \left(M + \frac{I_{CM}}{a^2}\right)s^2 + Cs + K$

→ Transfer function: $H(s) = \frac{1}{\left(M + \frac{I_{CM}}{a^2}\right)s^2 + Cs + K} = \frac{X_d(s)}{F(s)}$ for zero I.C.s

Frequency Response Functions

To obtain the frequency response function, the transfer function is evaluated at *s=jω*

$$\rightarrow \quad H(s)\big|_{s=j\omega} = H(j\omega) = \frac{X_d(j\omega)}{F(j\omega)} = \frac{1}{K - \left(M + \frac{I_{CM}}{a^2}\right)\omega^2 + j\omega C}$$

Then the amplitude and phase are found by calculating the modulus and argument of this complex function of *ω*:

$$\rightarrow \quad \left\| H(j\omega) \right\| = \frac{1}{\sqrt{\left[K - \left(M + \frac{I_{CM}}{a^2}\right)\omega^2\right]^2 + \left[\omega C\right]^2}} = \frac{1/K}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\varsigma \frac{\omega}{\omega_n}\right]^2}}$$

$$\rightarrow \quad \angle H(j\omega) = -\tan^{-1} \frac{\omega C}{K - \left(M + \frac{I_{CM}}{a^2}\right)\omega^2} = -\tan^{-1} \frac{2\varsigma \frac{\omega}{\omega_n}}{1 - \left(\frac{\omega}{\omega_n}\right)^2}$$

Sinusoidal Solution

So in conclusion we can state the following:

→

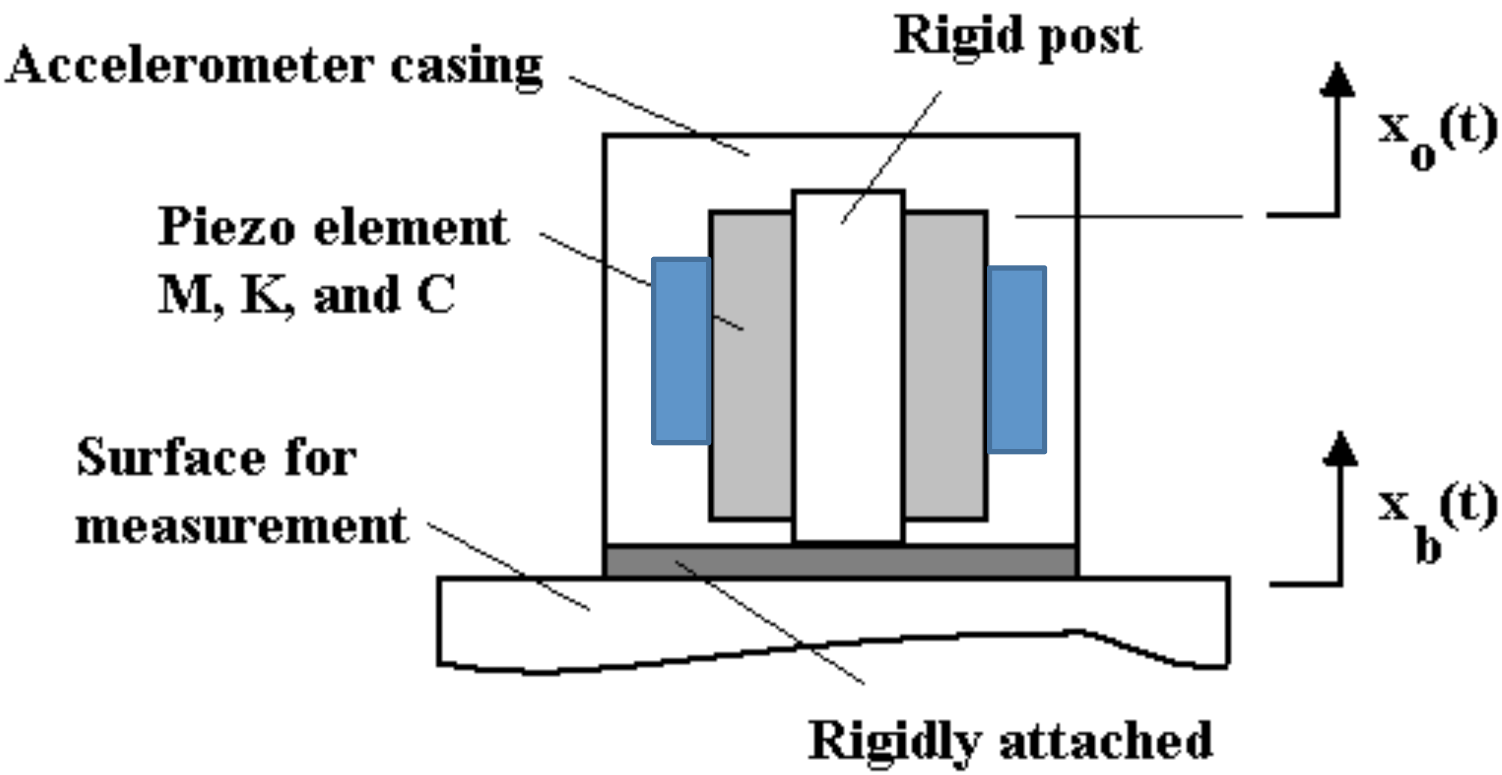
$$f(t) = F_i \cos(\omega t + \phi_i) \text{ and } x_p(t) = F_i \left\| H(j\omega) \right\| \cos(\omega t + \phi_i + \angle H(j\omega))$$

In words, a <u>simple-harmonic</u> <u>sinusoidal</u> excitation force applied to a <u>linear</u> <u>time-invariant</u> system produces a simple-harmonic sinusoidal response in the <u>steady state</u> at a different amplitude and phase than the excitation force.

→

Keywords: linear, time-invariant, sinusoidal, steady state, amplitude, phase

Example (Accelerometer)



$$x_o(t) - x_b(t) \sim \ddot{x}_b(t)$$

Example (Accelerometer)

→ $$M\ddot{z} + C\dot{z} + Kz = -M\ddot{x}_b$$ where $z(t)=x_o(t)-x_b(t)$

→ $$x_b(t) = \sum_{n=0}^{\infty}\left[a_n \cos(n\omega_o t) + b_n \sin(n\omega_o t)\right]$$

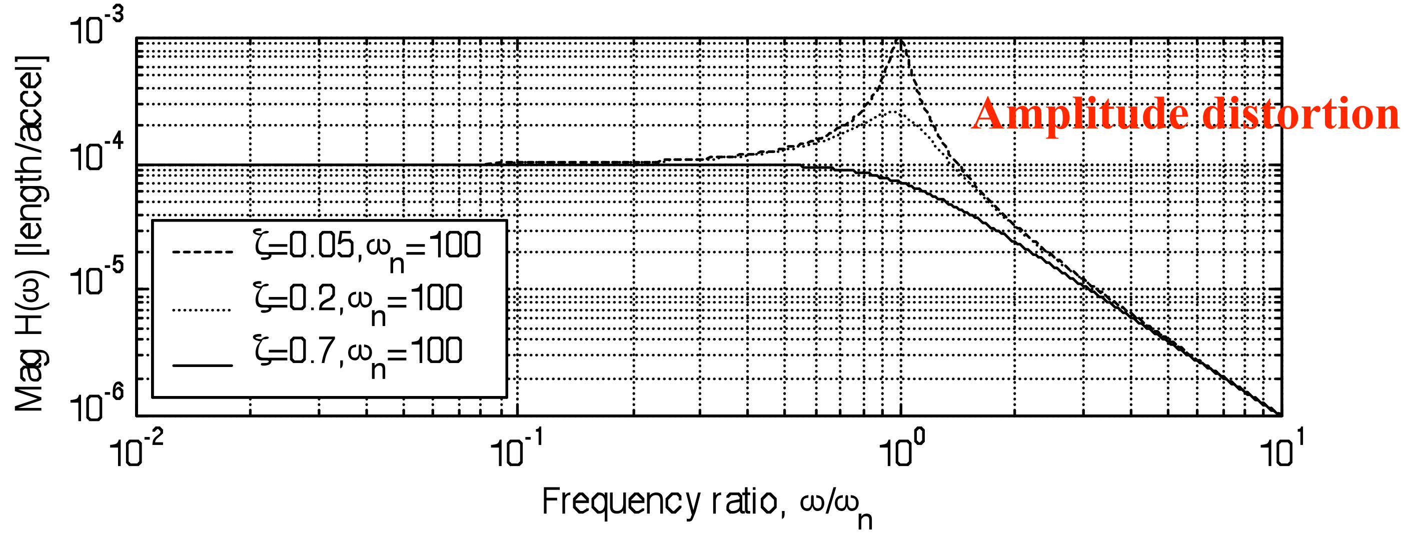
→ $$H(\omega) = \frac{Z(\omega)}{-\omega^2 X_b(\omega)} = \frac{-M}{K - M\omega^2 + j\omega C}$$

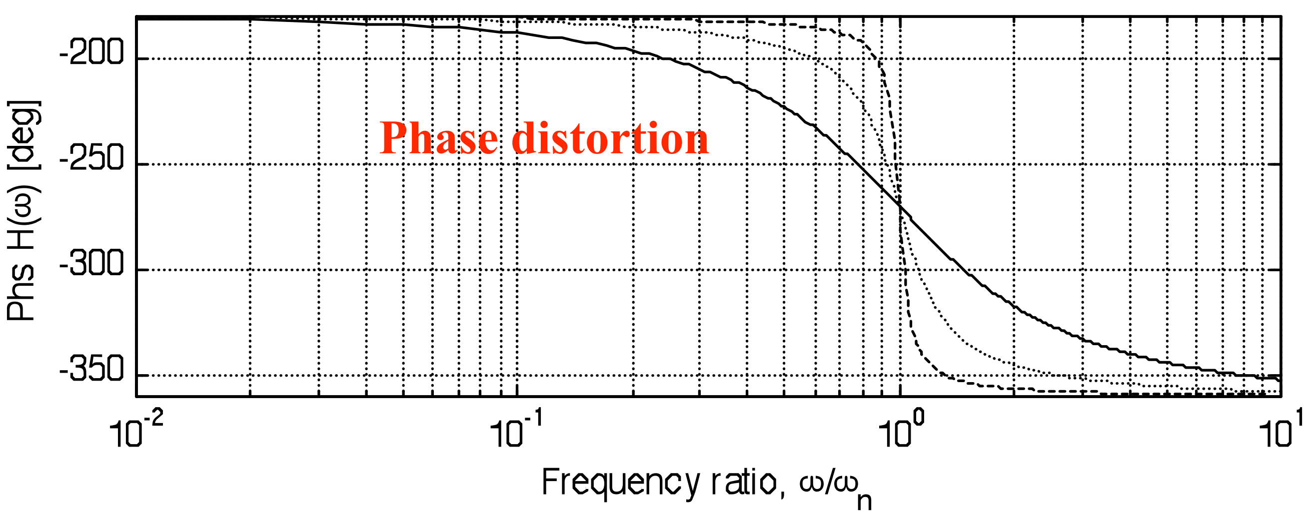
where $$\|H(\omega)\| = \frac{M}{\sqrt{\left[K - M\omega^2\right]^2 + \left[\omega C\right]^2}} = \frac{1/\omega_n^2}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\varsigma\frac{\omega}{\omega_n}\right]^2}}$$

and $$\angle H(\omega) = -180^o - \tan^{-1}\frac{\omega C}{K - M\omega^2} = -180^o - \tan^{-1}\frac{2\varsigma\frac{\omega}{\omega_n}}{1-\left(\frac{\omega}{\omega_n}\right)^2}$$

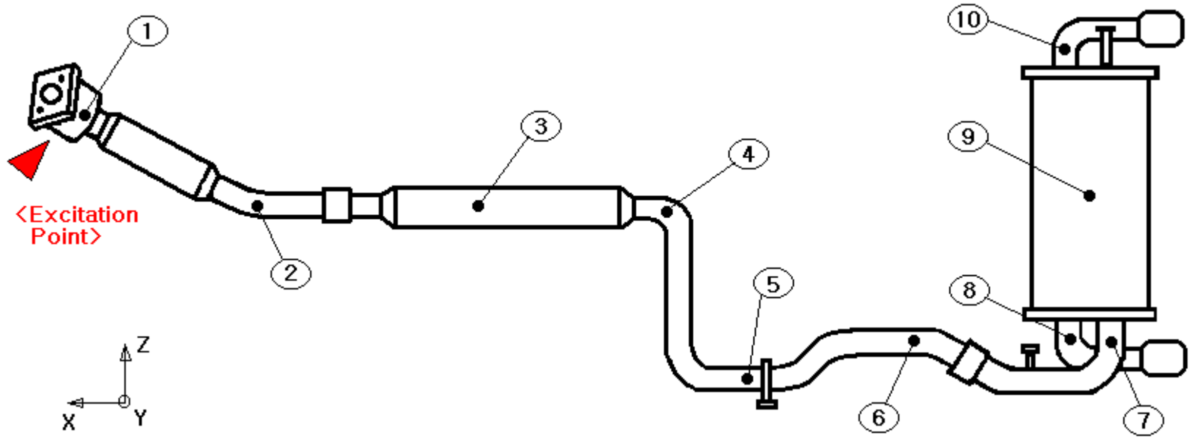
→ $$z_p(t) = \sum_{n=0}^{\infty}\|H(n\omega_o)\|\left[a_n \cos(n\omega_o t + \angle H(n\omega_o)) + b_n \sin(n\omega_o t + \angle H(n\omega_o))\right]$$

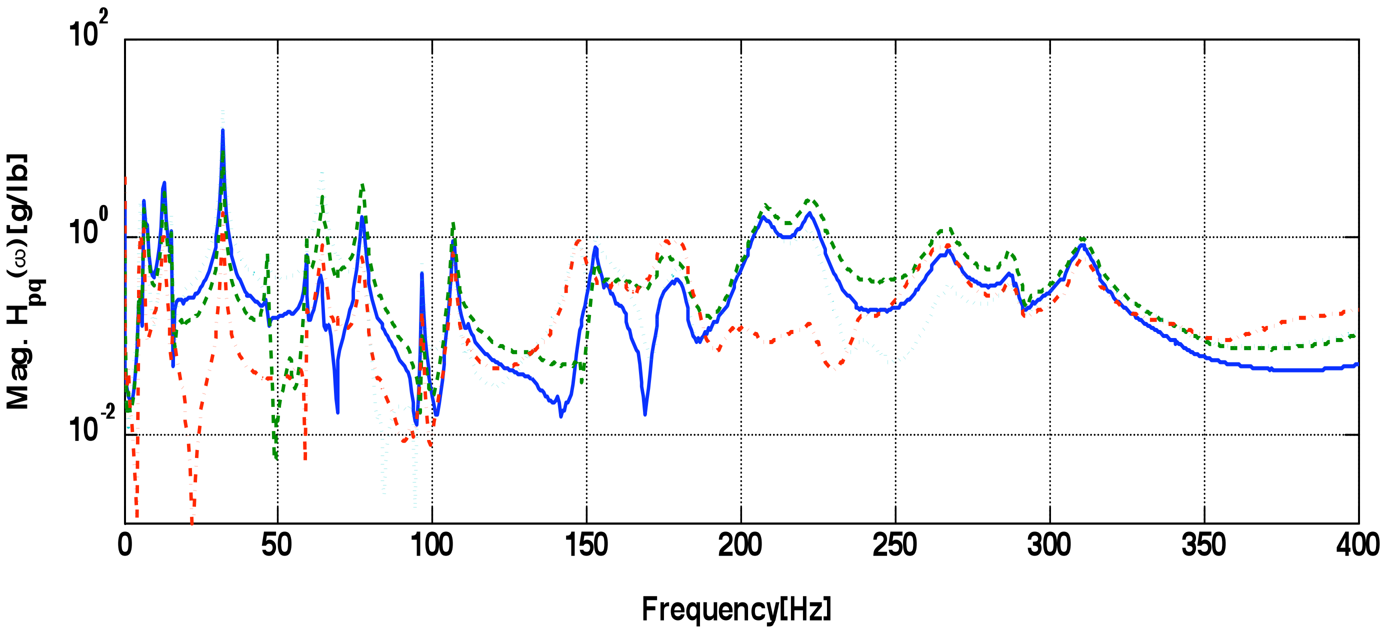
Example (Accelerometer)





Example (Exhaust system)





Frequency Response Functions

How do we get the velocity frequency response function or the acceleration frequency response function given?

$$\rightarrow \quad H(j\omega) = \frac{X_d(j\omega)}{F(j\omega)} = \frac{1}{K - \left(M + \frac{I_{CM}}{a^2}\right)\omega^2 + j\omega C}$$

Thinking back to the Laplace transform, we simply multiply by *jω*:

$$\rightarrow \quad H_v(j\omega) = \frac{V_d(j\omega)}{F(j\omega)} = \frac{j\omega}{K - \left(M + \frac{I_{CM}}{a^2}\right)\omega^2 + j\omega C}$$

$$\rightarrow \quad H_a(j\omega) = \frac{A_d(j\omega)}{F(j\omega)} = \frac{-\omega^2}{K - \left(M + \frac{I_{CM}}{a^2}\right)\omega^2 + j\omega C}$$