ME 563
Mechanical Vibrations
Lecture #16

Forced Response

(Step Input, Harmonic Excitation)

Free + Forced Response

Because the equations we are solving are linear in nature, we can simply add the free and forced response components to obtain the total (general) solution to the equations of motion:

$$\rightarrow M\ddot{x}_d + C\dot{x}_d + Kx_d = f(t)$$

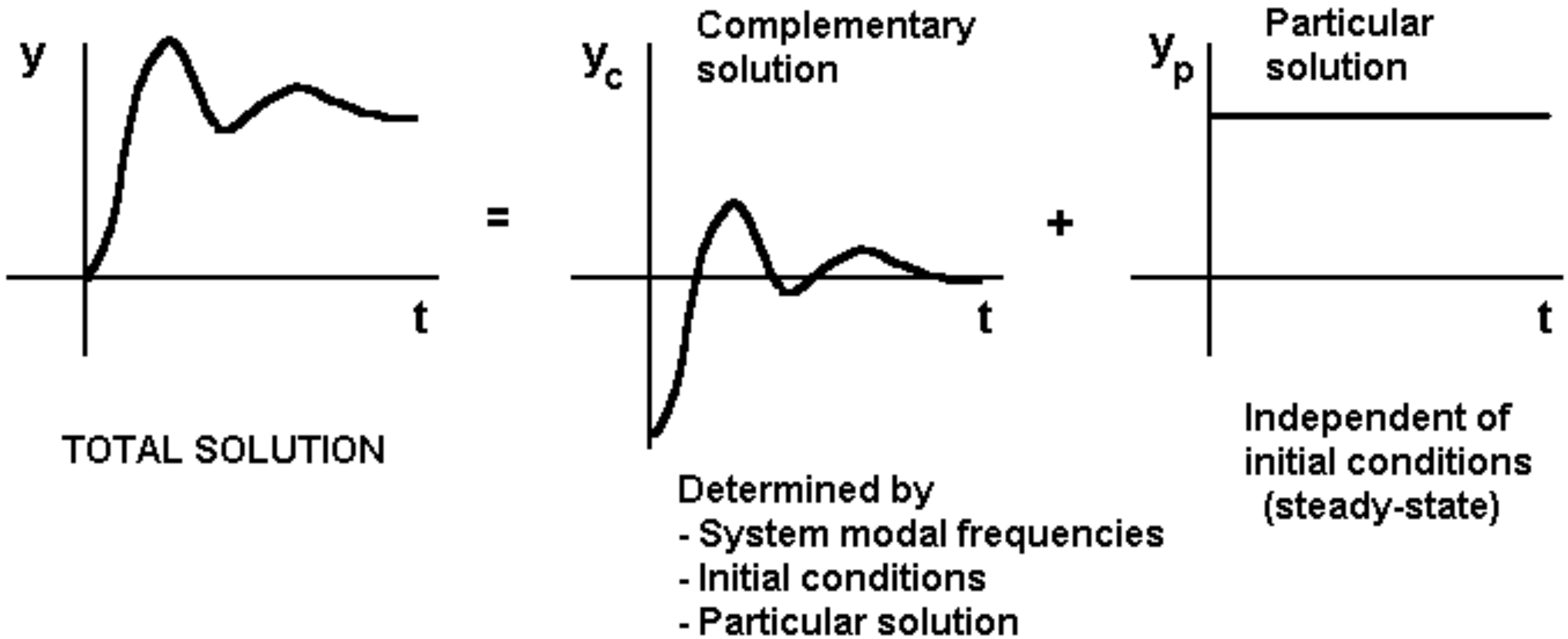
$$\rightarrow M\frac{d^2}{dt^2}(x_c + x_p) + C\frac{d}{dt}(x_c + x_p) + K(x_c + x_p) = f(t)$$

$$\rightarrow (M\ddot{x}_c + C\dot{x}_c + Kx_c) + (M\ddot{x}_p + C\dot{x}_p + Kx_p) = f(t)$$

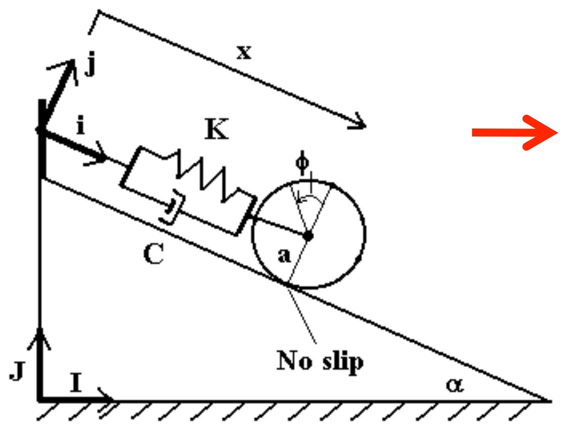
$$\rightarrow 0 + f(t) = f(t)$$

Now we will consider several different forms of *f(t)*.

Each Part of the Solution



Step Inputs



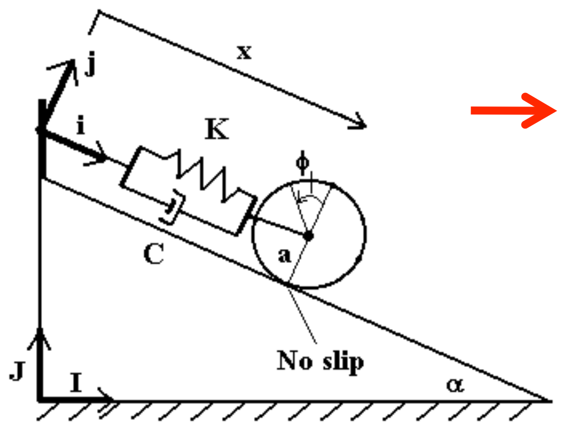
$$\left(M + \frac{I_{CM}}{a^2}\right)\ddot{x} + C\dot{x} + Kx = Mg\sin\alpha + Kx_u$$

If we release the disc on an incline from the position x=0, the response we obtain is the step response to a constant input.

$$X_o e^{\sigma t}\cos\left(\omega_d t + \phi_o\right) \quad + \quad \frac{Mg\sin\alpha}{K} + x_u$$

Complementary **Particular**

Step Inputs



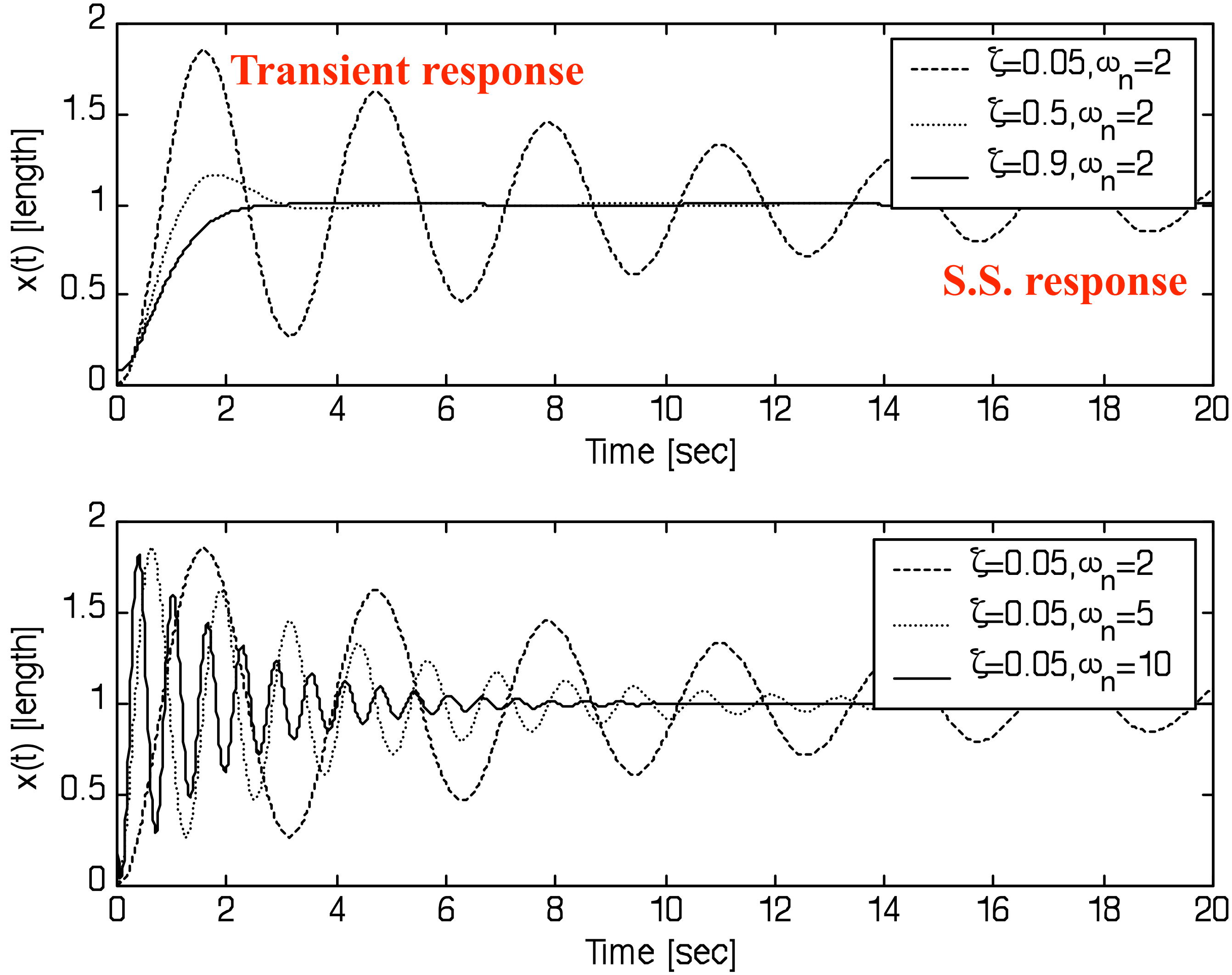
$$x(t) = X_o e^{\sigma t} \cos(\omega_d t + \phi_o) + \frac{Mg\sin\alpha}{K} + x_u$$

$$= \begin{bmatrix} \text{Transient} \\ \text{Solution} \end{bmatrix} + \begin{bmatrix} \text{Steady-state} \\ \text{Solution} \end{bmatrix}$$

After applying zero initial conditions, the following solution is obtained:

$$x(t) = \frac{Mg\sin\alpha + x_u}{K} \left[1 - \frac{\omega_n}{\omega_d} e^{\sigma t} \cos\left(\omega_d t - \tan^{-1} \frac{\varsigma}{\sqrt{1-\varsigma^2}} \right) \right] \quad \text{for } t > 0$$

Step Inputs


Transient response
S.S. response
ζ=0.05,ωn=2
ζ=0.5,ωn=2
ζ=0.9,ωn=2
x(t) [length]
Time [sec]
ζ=0.05,ωn=2
ζ=0.05,ωn=5
ζ=0.05,ωn=10
x(t) [length]
Time [sec]

Step Inputs

In MATLAB, we use the `step(sys)` command to calculate and plot the step response.

Harmonic Inputs

Every periodic input force, *f(t)*, can be described exactly over one period, *t=0* to *T* sec, using a Fourier series:

Real Fourier series

$$x_d(t) = \sum_{n=0}^{\infty} \left[a_n \cos(n\omega_o t) + b_n \sin(n\omega_o t) \right]$$

$$\text{where } a_n = \frac{2}{T}\int_{-T/2}^{T/2} x_d(t)\cos(n\omega_o t)\,dt$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} x_d(t)\sin(n\omega_o t)\,dt \text{ with } T = \frac{2\pi}{\omega_o}$$

Complex Fourier series

$$x_d(t) = \sum_{n=-\infty}^{\infty} c_n e^{jnw_o t}$$

$$\text{where } c_n = \frac{1}{T}\int_{-T/2}^{T/2} x_d(t) e^{jn\omega_o t}\,dt \text{ with } c_n = \frac{a_n}{2} - j\frac{b_n}{2} \text{ and } c_{-n} = c_n^* = \frac{a_n}{2} + j\frac{b_n}{2}$$

Therefore, we can calculate the steady state response to any periodic force using a sum of responses to sinusoidal forces.

Harmonic Inputs

If we consider a cosinusoidal input force:

$$\rightarrow \left(M + \frac{I_{CM}}{a^2}\right)\ddot{x}_d + C\dot{x}_d + Kx_d = F_i \cos(\omega t + \phi_i)$$

and assume the response is of the form, $x_p(t)=X_p cos(\omega t+\varphi_p)$ **then:**

$$\rightarrow \left[K - \left(M + \frac{I_{CM}}{a^2}\right)\omega^2\right]X_p \cos(\omega t + \phi_p) - \omega C X_p \sin(\omega t + \phi_p) = F_i \cos(\omega t + \phi_i)$$

$$\rightarrow \Rightarrow \left[K - \left(M + \frac{I_{CM}}{a^2}\right)\omega^2\right]X_p \cos\phi_p - \omega C X_p \sin\phi_p = F_i \cos\phi_i$$

$$\rightarrow \Rightarrow -\left[K - \left(M + \frac{I_{CM}}{a^2}\right)\omega^2\right]X_p \sin\phi_p - \omega C X_p \cos\phi_p = -F_i \sin\phi_i$$

Harmonic Inputs

If we consider a co-sinusoidal input force, $F_i \cos(\omega t + \phi_i)$ **:**

$$\rightarrow \left(M + \frac{I_{CM}}{a^2}\right)\ddot{x}_d + C\dot{x}_d + Kx_d = F_i \cos(\omega t + \phi_i)$$

then the relative amplitude and phase of the response are:

$$\rightarrow X_p = \frac{F_i}{\sqrt{\left[K - \left(M + \frac{I_{CM}}{a^2}\right)\omega^2\right]^2 + (\omega C)^2}}$$

$$\rightarrow \phi_p = \phi_i - \tan^{-1}\frac{\omega C}{K - \left(M + \frac{I_{CM}}{a^2}\right)\omega^2}$$

$$x_p(t) = X_p \cos(\omega t + \varphi_p)$$

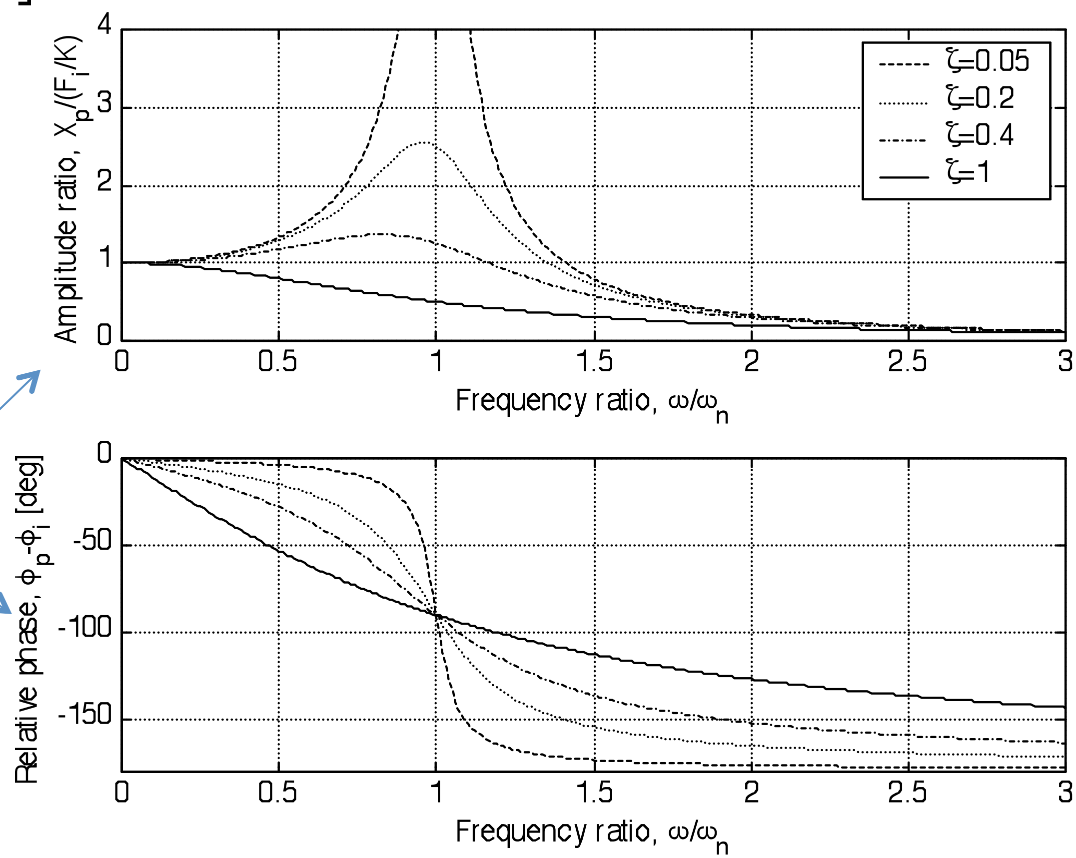
Frequency Response Functions

$$X_p = \frac{F_i / K}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\varsigma\frac{\omega}{\omega_n}\right]^2}}$$

Normalized expressions using modal parameters

$$\phi_p = \phi_i - \tan^{-1}\frac{2\varsigma\frac{\omega}{\omega_n}}{1-\left(\frac{\omega}{\omega_n}\right)^2}$$

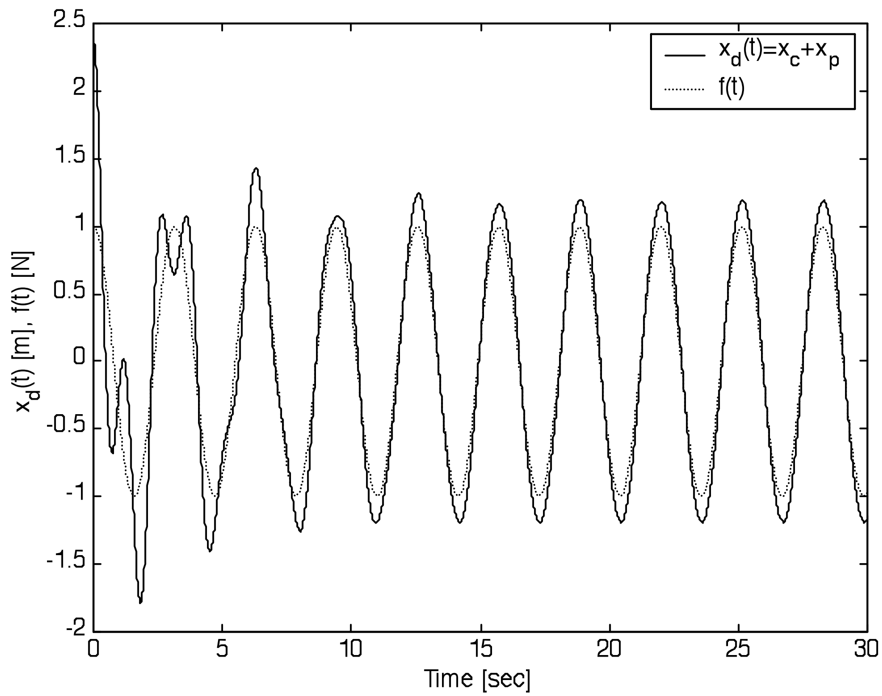
Sometimes called Bode Diagrams



Total Response

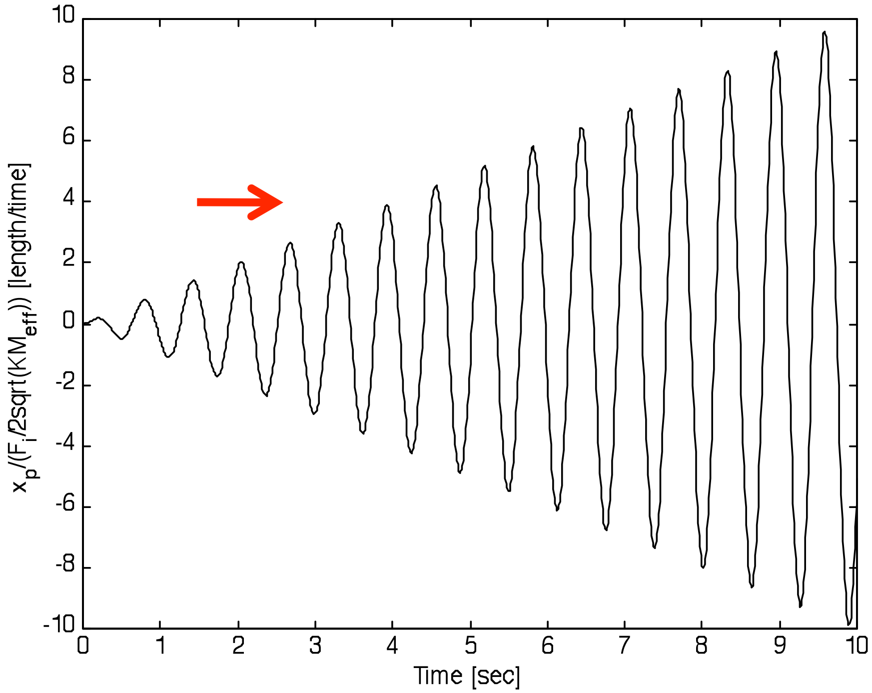
As with the step response, we now must add the steady state (particular) response to the free response to obtain the total response expression:

$$\rightarrow x_d(t) = X_o e^{\sigma t}\cos\left(\omega_d t + \phi_o\right) + X_p \cos\left(\omega t + \phi_p\right)$$



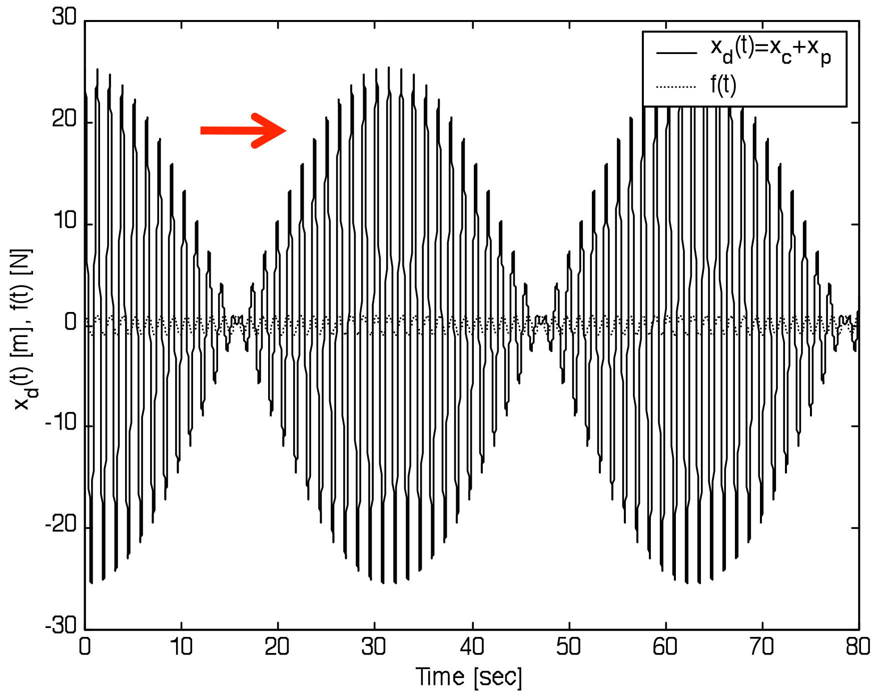
Some Examples (C=0)…

Resonant response of undamped system



$$x_p(t) = \frac{F_i}{2\sqrt{KM_{eff}}} t \sin(\omega_n t)$$

Response near a resonant frequency of vibration



$$x_1(t) + x_2(t) = A\sin(\omega t) + A\sin(\omega t + \Delta\omega t)$$

$$= 2A\cos\left(\frac{\Delta\omega t}{2}\right)\sin\left(\omega t + \frac{\Delta\omega t}{2}\right)$$