ME 563
Mechanical Vibrations
Lecture #15

Finite Element Approximations for Rods and Beams

Need for Finite Elements

Continuous system vibration equations of motion are appropriate for applications where the cross-sectional properties of the component are nearly constant. However, if the geometry or material properties change as a function of *x* (e.g., stepped shaft, cracked component, etc.), then a modeling technique is needed to describe these discontinuities.

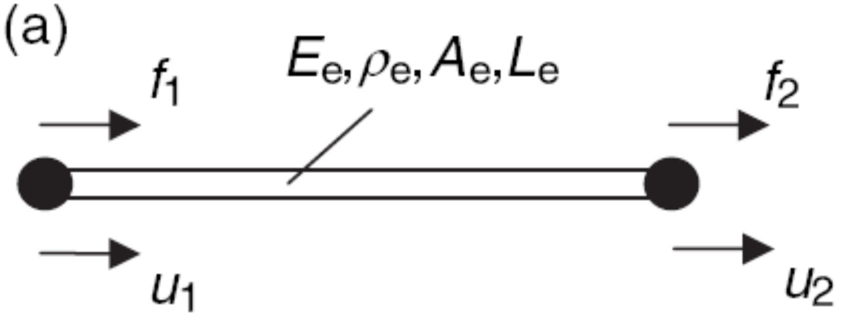
$$I_s(x)\frac{\partial^2\theta(x,t)}{\partial t^2}=\frac{\partial}{\partial x}\left(GJ(x)\frac{\partial\theta(x,t)}{\partial x}\right)$$

Separation of variables?

Finite element models are one means to describe such discontinuities. These models represent <u>continuous</u> systems using <u>discrete</u> (lumped parameter) models.

Rod Elements

Consider a rod undergoing longitudinal vibrations. If we wish to model the rod using lumped elements, how do we choose the mass and stiffness values?



(b)
$$\begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} = [K_e] \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} = [M_e] \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix}$$

We use energy methods:

$$\mathbf{KE_{model}=KE_{rod}} \qquad \mathbf{PE_{model}=PE_{rod}}$$

Rod Element Equations

The resulting force-motion relationships for a rod element are given by:

Elemental **stiffness matrix:**

→ $$\begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} = \begin{bmatrix} K_\mathrm{e} & -K_\mathrm{e} \\ -K_\mathrm{e} & K_\mathrm{e} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \text{where } K_\mathrm{e} = \frac{A_\mathrm{e} E_\mathrm{e}}{L_\mathrm{e}}$$

Elemental **mass matrix:**

→ $$\begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix}$$

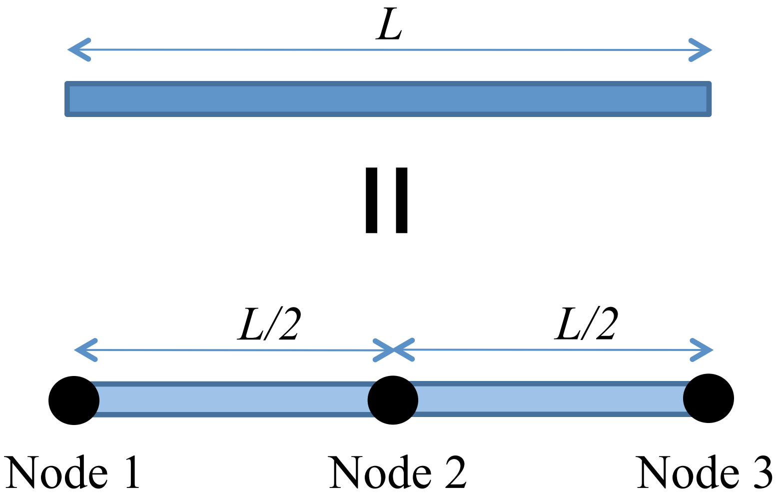
Per unit length!

$$\text{where } M_{11} = M_{22} = \frac{\rho_\mathrm{e} L_\mathrm{e}}{3} \text{ and } M_{12} = M_{21} = \frac{\rho_\mathrm{e} L_\mathrm{e}}{6}$$

Uniform Rod Example

Let's consider a rod with uniform cross-sectional properties as an example of how to utilize finite elements.

For a rod of total length *L*, cross-sectional area *A*, modulus *E*, and density per unit volume *ρ*, we can calculate the stiffness and mass lumped parameters for each element of the rod on the previous slide. We will use <u>2 elements</u> to describe the rod:



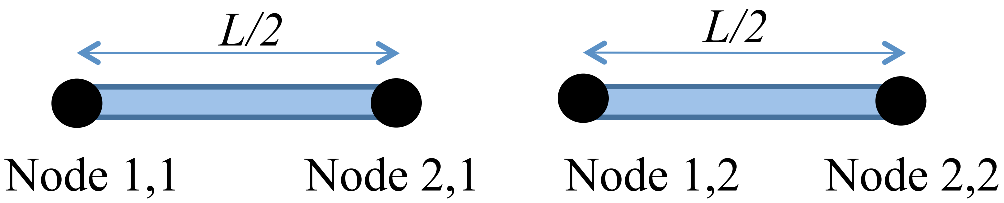
$$\rightarrow \left[K_e\right] = \frac{AE}{L/2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\rightarrow \left[M_e\right] = \rho AL/2\begin{bmatrix} 1/3 & 1/6 \\ 1/6 & 1/3 \end{bmatrix}$$

Per unit vol!

Uniform Rod Example

Now we must combine the elemental stiffness and mass matrices into global stiffness and mass matrices by enforcing constraints at each node of the model:



$$\vec{f}_{node\#2,element\#1} = -\vec{f}_{node\#1,element\#2}$$

$$\vec{u}_{node\#2,element\#1} = \vec{u}_{node\#1,element\#2}$$

Assembling Global Matrices

For example, the two element stiffness matrices are given by:

→ →

$$[K_1] = \frac{AE}{L/2}\begin{array}{cc} \text{N1} & \text{N2} \\ \left[\begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array}\right] \end{array}\begin{array}{c} \text{N1} \\ \text{N2} \end{array} \text{ and } [K_2] = \frac{AE}{L/2}\begin{array}{cc} \text{N1} & \text{N2} \\ \left[\begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array}\right] \end{array}\begin{array}{c} \text{N1} \\ \text{N2} \end{array}$$

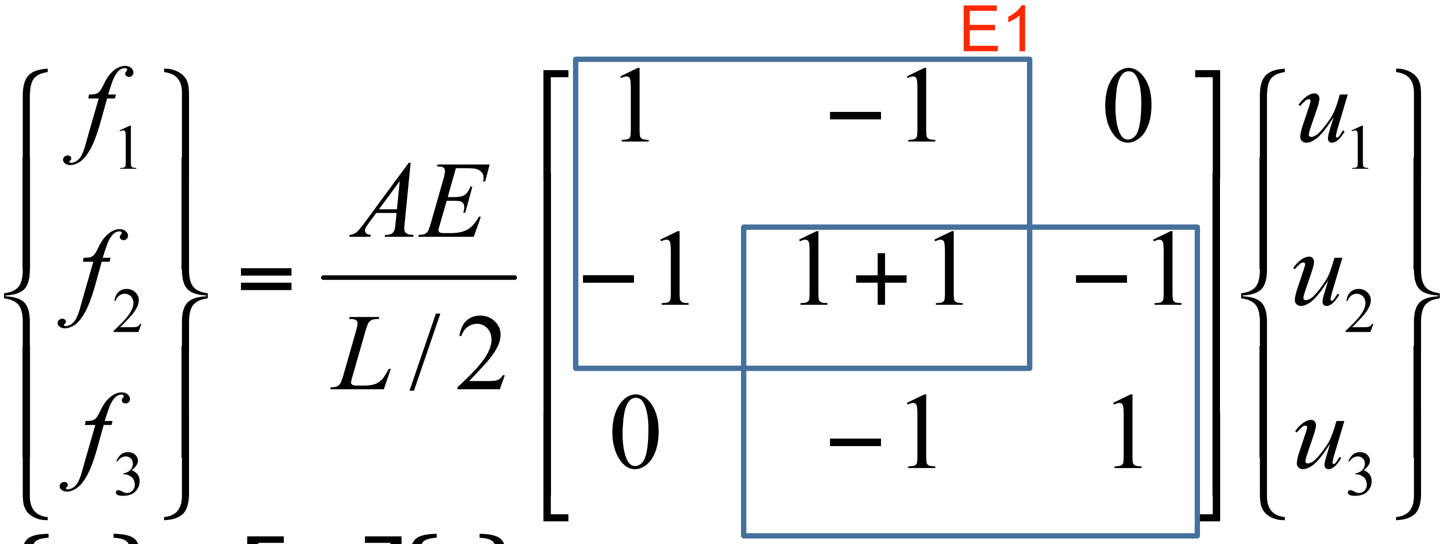
$$\begin{Bmatrix} f_{1,1} \\ f_{2,1} \end{Bmatrix} = [K_1]\begin{Bmatrix} u_{1,1} \\ u_{2,1} \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} f_{1,2} \\ f_{2,2} \end{Bmatrix} = [K_2]\begin{Bmatrix} u_{1,2} \\ u_{2,2} \end{Bmatrix}$$

So the constraints are enforced as follows:

→ $f_{2,1} = -f_{1,2}$ and $u_{2,1} = u_{1,2}$

Assembling Global Matrices

To enforce these two constraints (compatibility and continuity), we (a) add rows corresponding to nodes that are coincident to eliminate internal forces at those nodes, and (b) equate the motions at those nodes.



$$\begin{Bmatrix} f_1 \\ f_2 \\ f_3 \end{Bmatrix} = \frac{AE}{L/2}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1+1 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$\{f\} = [K]\{u\}$$

Assembling Global Matrices

The resulting global stiffness matrix is:

$$[K] = \frac{AE}{L/2}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

Likewise, the resulting global mass matrix is:

$$[M] = \rho AL/2\begin{bmatrix} 1/3 & 1/6 & 0 \\ 1/6 & 2/3 & 1/6 \\ 0 & 1/6 & 1/3 \end{bmatrix}$$

System Equations of Motion

We can now write the complete set of equations of motion:

$$\rho AL/2\begin{bmatrix} \overset{\longrightarrow}{1/3} & 1/6 & 0 \\ 1/6 & 2/3 & 1/6 \\ 0 & 1/6 & 1/3 \end{bmatrix}\begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \\ \ddot{u}_3 \end{Bmatrix} + \frac{AE}{L/2}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \end{Bmatrix}$$

Now let's compare the modal frequencies and modal vectors (shapes) obtained using this finite element model and the continuous system model.

We solve the eigenvalue problems (discrete and continuous) in the manner we discussed previously.

Comparison of ω_n

Assume the rod is of length 1 m, cross-sectional area 0.1 m, modulus 70 GPa, and density 2700 kg/m^3.

Discrete model gives resonant frequencies (`eig` command),

```
1.0e+004 *

          0 + 3.5277i
      →   0 + 1.7638i
          0 + 0.0000i
```

Continuous model gives,

$$\omega_n = n\pi\sqrt{\frac{E}{\rho L^2}}\left(\sqrt{\frac{N/m^2}{kg/m^3 * m^2}}\right) \text{ for integer } n = 0,1,2$$

```
1.0e+004 *
          0
  →  1.5996
     3.1992
```

Comparison of ω_n

Generally speaking, finite element model estimates of natural frequencies are on the high side because the approximation of lumped stiffness introduces a constraint.

If we add another element for a total of 3, then ω_n are:

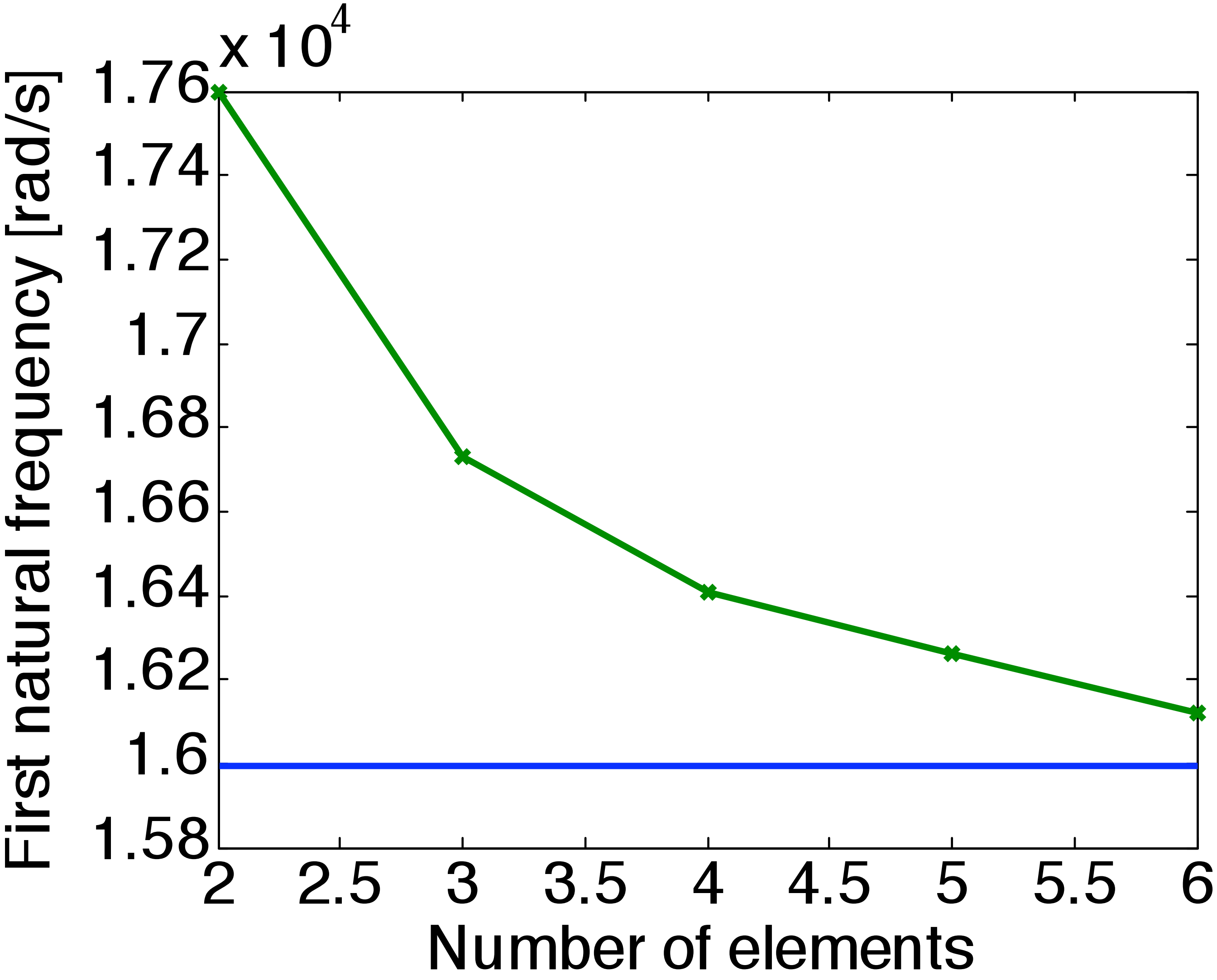
```
1.0e+004 *
        0 + 5.2915i
        0 + 3.7417i
     →  0 + 1.6733i
        0 + 0.0000i
```

Continuous model gives,

$$\omega_n = n\pi\sqrt{\frac{E}{\rho L^2}}\left(\sqrt{\frac{N/m^2}{kg/m^3 * m^2}}\right) \text{ for integer } n = 0,1,2$$

```
1.0e+004 *
            0
   →   1.5996
       3.1992
       4.7989
```

Comparison of ω_n

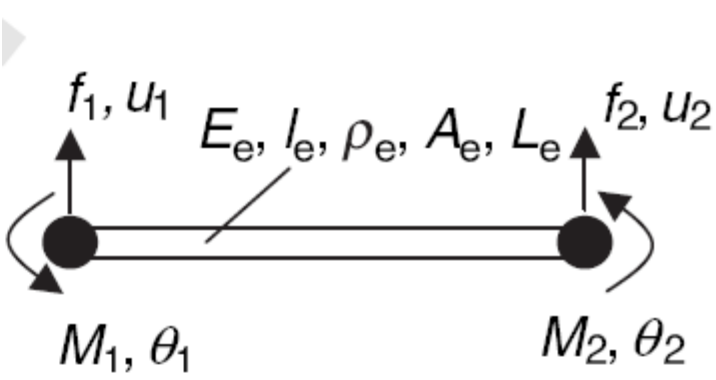


```
% Compare wn's for rod
% finite element model and continuous model

% Two elements
A=0.1; E=70e9; L=1; rho=2700; K=A*E/(L/2)*[1 -1 0;-1 2 -1;0 -1 1]; M=rho*(L/2)*A*[1/3 1/6 0;1/6 2/3 1/6;0 1/6 1/3];

disp('2');
sqrt(-eig(inv(M)*K))

[0; 1*pi*sqrt(E/(rho*L^2)); 2*pi*sqrt(E/(rho*L^2))]

% Three elements
A=0.1; E=70e9; L=1; rho=2700; K=A*E/(L/3)*[1 -1 0 0;-1 2 -1 0;0 -1 2 -1;0 0 -1 1]; M=rho*(L/3)*A*[1/3 1/6 0 0;1/6 2/3 1/6 0;0 1/6 2/3 1/6;0 0 1/6 1/3];

disp('3');
sqrt(-eig(inv(M)*K))

[0; 1*pi*sqrt(E/(rho*L^2)); 2*pi*sqrt(E/(rho*L^2)); 3*pi*sqrt(E/(rho*L^2))]

% Four elements
A=0.1; E=70e9; L=1; rho=2700; K=A*E/(L/4)*[1 -1 0 0 0;-1 2 -1 0 0;0 -1 2 -1 0;0 0 -1 2 -1;0 0 0 -1 1]; M=rho*(L/4)*A*[1/3 1/6 0 0 0;1/6 2/3 1/6 0 0;0 1/6 2/3 1/6 0;0 0 1/6 2/3 1/6;0
0 0 1/6 1/3];

disp('4');
sort(sqrt(-eig(inv(M)*K)))

[0; 1*pi*sqrt(E/(rho*L^2)); 2*pi*sqrt(E/(rho*L^2)); 3*pi*sqrt(E/(rho*L^2));  4*pi*sqrt(E/(rho*L^2))]

% Five elements
A=0.1; E=70e9; L=1; rho=2700; K=A*E/(L/5)*[1 -1 0 0 0 0;-1 2 -1 0 0 0;0 -1 2 -1 0 0;0 0 -1 2 -1 0;0 0 0 -1 2 -1;0 0 0 0 -1 1]; M=rho*(L/5)*A*[1/3 1/6 0 0 0 0;1/6 2/3 1/6 0 0 0;0 1/6
2/3 1/6 0 0;0 0 1/6 2/3 1/6 0;0 0 0 1/6 2/3 1/6;0 0 0 0 1/6 1/3];

disp('5');
sort(sqrt(-eig(inv(M)*K)))

[0; 1*pi*sqrt(E/(rho*L^2)); 2*pi*sqrt(E/(rho*L^2)); 3*pi*sqrt(E/(rho*L^2));  4*pi*sqrt(E/(rho*L^2)); 5*pi*sqrt(E/(rho*L^2))]

% Six elements
A=0.1; E=70e9; L=1; rho=2700; K=A*E/(L/6)*[1 -1 0 0 0 0 0;-1 2 -1 0 0 0 0;0 -1 2 -1 0 0 0;0 0 -1 2 -1 0 0;0 0 0 -1 2 -1 0;0 0 0 0 -1 2 -1; 0 0 0 0 0 -1 1]; M=rho*(L/6)*A*[1/3 1/6 0 0
0 0 0;1/6 2/3 1/6 0 0 0 0;0 1/6 2/3 1/6 0 0 0;0 0 1/6 2/3 1/6 0 0;0 0 0 1/6 2/3 1/6 0;0 0 0 0 1/6 2/3 1/6;0 0 0 0 0 1/6 1/3];

disp('6');
sort(sqrt(-eig(inv(M)*K)))

[0; 1*pi*sqrt(E/(rho*L^2)); 2*pi*sqrt(E/(rho*L^2)); 3*pi*sqrt(E/(rho*L^2));  4*pi*sqrt(E/(rho*L^2)); 5*pi*sqrt(E/(rho*L^2))]

figure(1);
plot(2:6,[1.5996e4 1.5996e4 1.5996e4 1.5996e4 1.5996e4],2:6,[1.76e4 1.673e4 1.641e4 1.626e4 1.612e4],'x-');
xlabel('Number of elements');
ylabel('First natural frequency [rad/s]');
```

Beam Element Equations

For a beam that is dominated by bending effects and obeys the Bernoulli-Euler assumptions of small deflections, plane sections remain plane, and constant length neutral axis, we define a four degree of freedom finite element as follows:

(a)



(b)

$$\begin{Bmatrix} f_1 \\ M_1 \\ f_2 \\ M_2 \end{Bmatrix} = [K_e] \begin{Bmatrix} u_1 \\ \theta_1 \\ u_2 \\ \theta_2 \end{Bmatrix}$$

$$\begin{Bmatrix} f_1 \\ M_1 \\ f_2 \\ M_2 \end{Bmatrix} = [M_e] \begin{Bmatrix} \ddot{u}_1 \\ \ddot{\theta}_1 \\ \ddot{u}_2 \\ \ddot{\theta}_2 \end{Bmatrix}$$

Beam Element Equations

The elemental equations for the beam are given by:

Elemental **stiffness matrix:**

$$\begin{Bmatrix} f_1 \\ M_1 \\ f_2 \\ M_2 \end{Bmatrix} = \frac{E_e I_e}{L_e^3} \begin{bmatrix} 12 & 6L_e & -12 & 6L_e \\ 6L_e & 4L_e^2 & -6L_e & 2L_e^2 \\ -12 & -6L_e & 12 & -6L_e \\ 6L_e & 2L_e^2 & -6L_e & 4L_e^2 \end{bmatrix} \begin{Bmatrix} u_1 \\ \theta_1 \\ u_2 \\ \theta_2 \end{Bmatrix}$$

Elemental **mass matrix:**

Per unit length!

$$\begin{Bmatrix} f_1 \\ M_1 \\ f_2 \\ M_2 \end{Bmatrix} = \frac{\rho_e L_e}{420} \begin{bmatrix} 156 & 22L_e & 54 & -13L_e \\ 22L_e & 4L_e^2 & 13L_e & -3L_e^2 \\ 54 & 13L_e & 156 & -22L_e \\ -13L_e & -3L_e^2 & -22L_e & 4L_e^2 \end{bmatrix} \begin{Bmatrix} \ddot{u}_1 \\ \ddot{\theta}_1 \\ \ddot{u}_2 \\ \ddot{\theta}_2 \end{Bmatrix}$$