ME 563
Mechanical Vibrations
Lecture #14

Eigenvalue Problem for Continuous Systems

Discrete vs. Continuous

When we solved for the free response of a lumped parameter (discrete) mechanical system, we found that:

$$\rightarrow \quad \{\mathbf{x}(t)\} = [\Psi]\{\mathbf{p}(t)\}$$

Spatial pattern ($[\Psi]$) **Temporal pattern** ($\{\mathbf{p}(t)\}$)

But in a continuous system, there are an infinite number of possible spatial coordinates (and modal parameters). Therefore, we can expect to see a longer summation, but the form of the solution should still be of the form:

$$\rightarrow \quad y(x,t) = Y(x) \cdot G(t)$$

String Solution

Consider the string equation of motion:

$$\rightarrow \quad \frac{\partial^2 y(x,t)}{\partial t^2} - \frac{T}{\rho}\frac{\partial^2 y(x,t)}{\partial x^2} = 0$$

If we assume a solution of the form, $y(x,t) = Y(x)\cdot G(t)$ **, we obtain:**

$$\rightarrow \quad \frac{\partial^2 \left[Y(x)G(t)\right]}{\partial t^2} - \frac{T}{\rho}\frac{\partial^2 \left[Y(x)G(t)\right]}{\partial x^2} = 0$$

$$\rightarrow \quad Y(x)\frac{d^2 G(t)}{dt^2} - G(t)\frac{T}{\rho}\frac{d^2 Y(x)}{dx^2} = 0$$

$$\rightarrow \quad \frac{T}{\rho}\frac{1}{Y(x)}\frac{d^2 Y(x)}{dx^2} = \frac{1}{G(t)}\frac{d^2 G(t)}{dt^2}$$

String Solution

This last equation can only be satisfied for all *t* and *x* if:

$$\rightarrow \quad \frac{T}{\rho}\frac{1}{Y(x)}\frac{d^2Y(x)}{dx^2} = \frac{1}{G(t)}\frac{d^2G(t)}{dt^2} = \text{Constant}$$

We make an inspired guess for this constant (based on the second expression in the equation above):

$$\frac{T}{\rho}\frac{1}{Y(x)}\frac{d^2Y(x)}{dx^2} = -\omega_n^2$$

$$\rightarrow$$

$$\frac{1}{G(t)}\frac{d^2G(t)}{dt^2} = -\omega_n^2$$

String Solution

Now we can solve each of these equations independently in space and time (in that order):

$$\rightarrow \quad \frac{T}{\rho}\frac{1}{Y(x)}\frac{d^2Y(x)}{dx^2} = -\omega_n^2$$

$$\rightarrow \quad \frac{d^2Y(x)}{dx^2} + \frac{\rho\omega_n^2}{T}Y(x) = 0$$

$$\rightarrow \quad Y'' + \frac{\rho\omega_n^2}{T}Y = 0$$

This equation is of second order just like for the single degree of freedom system. Therefore, we can solve it given two boundary conditions on *Y(x)*.

Boundary Conditions

Consider the case where the string is fixed on both ends:
(why is this called the eigenvalue problem for the string?)

$$\rightarrow \quad Y'' + \frac{\rho\omega_n^2}{T} Y = 0 \text{ where } Y(0) = 0 = Y(L)$$

Then

$$\rightarrow \quad Y(x) = A_1 \sin\left(\omega_n \sqrt{\frac{\rho}{T}} x\right) + A_2 \cos\left(\omega_n \sqrt{\frac{\rho}{T}} x\right) \text{ for } 0 < x < L$$

$$\left.\begin{aligned} &\rightarrow \quad Y(0) = A_2 = 0 \\ &\rightarrow \quad Y(L) = A_1 \sin\left(\omega_n \sqrt{\frac{\rho}{T}} L\right) = 0 \end{aligned}\right\} \Rightarrow \text{non-trivial solutions, } \sin\left(\omega_n \sqrt{\frac{\rho}{T}} L\right) = 0$$

$$\rightarrow \quad \text{Natural frequencies, } \omega_n = n\pi \sqrt{\frac{T}{\rho L^2}} \text{ for integer } n = 1, 2, \cdots$$

Total Solution

The time domain equation can then be solved:

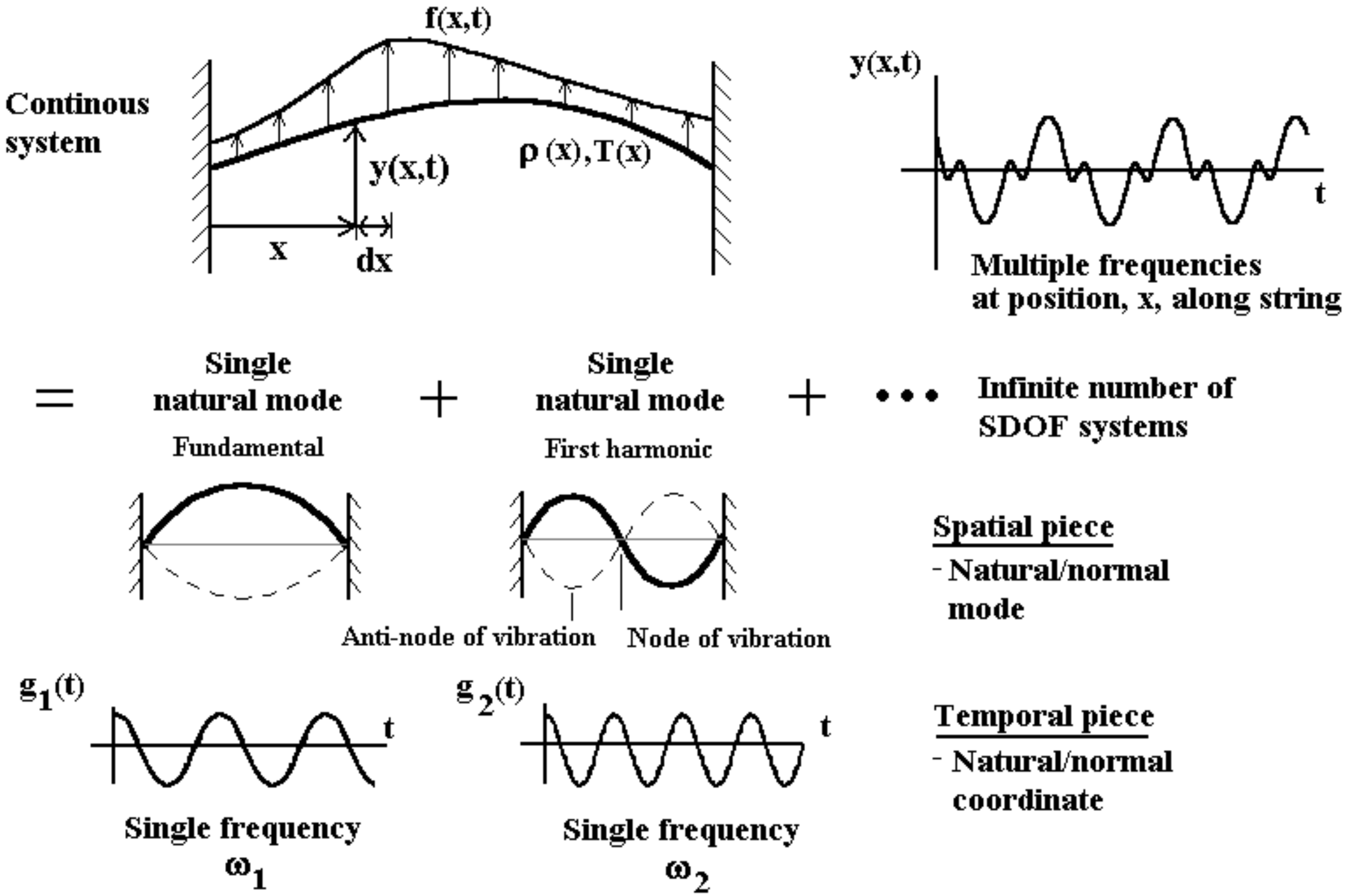
$$\rightarrow \quad \frac{1}{G(t)}\frac{d^2G(t)}{dt^2} = -\omega_n^2$$

$$\rightarrow \quad G(t) = G_n \cos\left(\omega_n t + \phi_n\right)$$

Then we can put the temporal and spatial components of the solution together:

$$\rightarrow \quad y(x,t) = \sum_{n=1}^{\infty} Y_n(x) G_n(t) = \sum_{n=1}^{\infty} Y_n(x) G_n \cos\left(\omega_n t + \phi_n\right)$$

Visualizing the Solution



Normal Coordinates

If we normalize the spatial functions, $Y_n(x)$, as follows:

→ $$\int_0^L \rho(x) Y_r(x) Y_s(x)\, dx = \delta_{rs} = \begin{cases} 0 & \text{for } r \neq s \\ 1 & \text{for } r = s \end{cases}$$

→ $$\int_0^L T(x) \frac{dY_r(x)}{dx} \frac{dY_s(x)}{dx} dx = \omega_r^2 \delta_{rs}$$

then we can expand the solution as follows:

→ $$y(x,t) = \sum_{n=1}^{\infty} Y_n(x) G_n(t) = \sum_{n=1}^{\infty} Y_n(x) G_n \cos(\omega_n t + \phi_n)$$

and multiply by one function $Y_r(x)$ at a time to uncouple them. We will do this later to obtain the forced response as well.

Uncoupled Normal Equations

$$\frac{\partial^2 \sum_n Y_n G_n}{\partial t^2} - \frac{T}{\rho}\frac{\partial^2 \sum_n Y_n G_n}{\partial x^2} = 0$$

$$\sum_n Y_n \ddot{G}_n - \frac{T}{\rho}\sum_n Y_n'' G_n = 0$$

$$Y_r\left(\sum_n Y_n \ddot{G}_n - \frac{T}{\rho}\sum_n Y_n'' G_n\right) = 0$$

$$\int_0^L \sum_n Y_r Y_n \ddot{G}_n dx - \frac{T}{\rho}\int_0^L \sum_n Y_r Y_n'' G_n dx = 0$$

$$\ddot{G}_r + \omega_r^2 G_r = 0$$

Uncoupled equations!

Beam Solution

Consider the beam equation of motion:

$$\rightarrow \quad m\frac{\partial^2 y(x,t)}{\partial t^2} = -EI\frac{\partial^2}{\partial x^2}\left(\frac{\partial^2 y(x,t)}{\partial x^2}\right)$$

If we assume a solution of the form, $y(x,t) = Y(x)\cdot G(t)$ **, we obtain:**

$$\rightarrow \quad \frac{d^2G(t)}{dt^2} + \omega_n^2 G(t) = 0$$

$$\rightarrow \quad \frac{d^4Y(x)}{dx^4} - \omega_n^2\frac{m}{EI}Y(x) = 0 \text{ with } Y(0) = 0 = Y(L),\ Y''(x)\big|_{x=0} = 0 = Y''(x)\big|_{x=L}$$

For a beam that is simply supported on both ends (because that leads to the simplest solution for the natural frequencies).

Beam Solution

The characteristic equation is obtained as for the beam:

$$\left(\frac{d^4}{dx^4} - \omega_n^2 \frac{m}{EI}\right) Y(x) = 0$$

which is written in operator notation to stress the fact that this is an eigenvalue problem like in the discrete case.

The possible solution of this equation are as follows:

$$Y(x) = A_1 \sin\left(\sqrt[4]{\omega_n{}^2 \frac{m}{EI}}x\right) + A_2 \cos\left(\sqrt[4]{\omega_n{}^2 \frac{m}{EI}}x\right) + A_3 \sinh\left(\sqrt[4]{\omega_n{}^2 \frac{m}{EI}}x\right) + A_4 \cosh\left(\sqrt[4]{\omega_n{}^2 \frac{m}{EI}}x\right)$$

Then the boundary conditions must be applied (next slide).

Boundary Conditions

- Recall $\sinh(x) = \dfrac{e^{x} - e^{-x}}{2}$ and $\cosh(x) = \dfrac{e^{x} + e^{-x}}{2}$
- $Y(0) = A_2 + A_4$ and $\left.\dfrac{d^2Y}{dx^2}\right|_{x=0} = \sqrt[2]{\omega_n^{\ 2}\dfrac{m}{EI}}\left(-A_2 + A_4\right) = 0$
- $\Rightarrow A_2 = 0 = A_4$
- $Y(x) = A_1 \sin\left(\sqrt[4]{\omega_n^{\ 2}\dfrac{m}{EI}}x\right) + A_3 \sinh\left(\sqrt[4]{\omega_n^{\ 2}\dfrac{m}{EI}}x\right)$
- $Y(L) = A_1 \sin\left(\sqrt[4]{\omega_n^{\ 2}\dfrac{m}{EI}}L\right) + A_3 \sinh\left(\sqrt[4]{\omega_n^{\ 2}\dfrac{m}{EI}}L\right) = 0$
- $\left.\dfrac{d^2Y}{dx^2}\right|_{x=L} = \sqrt[2]{\omega_n^{\ 2}\dfrac{m}{EI}}\left(-A_1 \sin\left(\sqrt[4]{\omega_n^{\ 2}\dfrac{m}{EI}}L\right) + A_3 \sinh\left(\sqrt[4]{\omega_n^{\ 2}\dfrac{m}{EI}}L\right)\right) = 0$
- $\Rightarrow A_3 = 0,\ \sin\left(\sqrt[4]{\omega_n^{\ 2}\dfrac{m}{EI}}L\right) = 0 \Rightarrow \sqrt[4]{\omega_n^{\ 2}\dfrac{m}{EI}}L = n\pi$ for $n = 1,2,\cdots$

Boundary Conditions

These equations can also be written and solved in matrix form:

$$
\begin{bmatrix}
0 & 1 & 0 & 1 \\
0 & -\sqrt[2]{\omega_n^{\,2}\frac{m}{EI}} & 0 & \sqrt[2]{\omega_n^{\,2}\frac{m}{EI}} \\
\sin\left(\sqrt[4]{\omega_n^{\,2}\frac{m}{EI}}L\right) & 0 & \sinh\left(\sqrt[4]{\omega_n^{\,2}\frac{m}{EI}}L\right) & 0 \\
-\sqrt[2]{\omega_n^{\,2}\frac{m}{EI}}\sin\left(\sqrt[4]{\omega_n^{\,2}\frac{m}{EI}}L\right) & 0 & \sqrt[2]{\omega_n^{\,2}\frac{m}{EI}}\sinh\left(\sqrt[4]{\omega_n^{\,2}\frac{m}{EI}}L\right) & 0
\end{bmatrix}
\begin{Bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{Bmatrix}
=
\begin{Bmatrix} Y(0) \\ 0 \\ Y(L) \\ 0 \end{Bmatrix}
$$