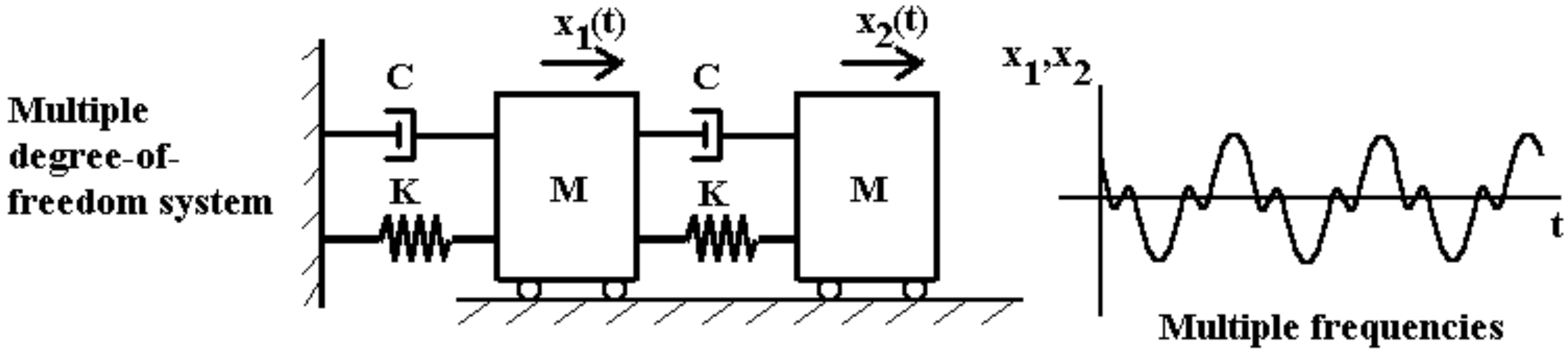
ME 563
Mechanical Vibrations
Lecture #13

Multiple Degree of Freedom

Modal Coordinate Transformation

What Did We Learn?



By making the assumption,*{x(t)}={A}e^st^*, we observed that the free response was actually a sum of the modes of vibration:

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = X_1 \begin{Bmatrix} \psi_{11} \\ \psi_{21} \end{Bmatrix} e^{\sigma_1 t} \cos(\omega_1 t + \phi_1) + X_2 \begin{Bmatrix} \psi_{12} \\ \psi_{22} \end{Bmatrix} e^{\sigma_2 t} \cos(\omega_2 t + \phi_2)$$

If we write this differently, we reveal the most important thing about this course:

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{bmatrix} \begin{Bmatrix} \psi_{11} \\ \psi_{21} \end{Bmatrix} & \begin{Bmatrix} \psi_{12} \\ \psi_{22} \end{Bmatrix} \end{bmatrix} \begin{Bmatrix} X_1 e^{\sigma_1 t} \cos(\omega_1 t + \phi_1) \\ X_2 e^{\sigma_2 t} \cos(\omega_2 t + \phi_2) \end{Bmatrix}$$

Coordinate Transformation

The solution for the free response is actually a coordinate transformation in disguise:

$$\rightarrow \{\mathbf{x}(t)\} = [\Psi]\{\mathbf{p}(t)\}$$

Generalized coordinates (Physical DOFs)

Principal coordinates (Modal DOFs)

Modal transformation matrix (Modal vectors)

If this is the form of the solution, why not use it to begin with?

Modal Solution Method

Consider the case again with no damping:

$$\begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 2K & -K \\ -K & K \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix} \begin{bmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{bmatrix} \begin{Bmatrix} \ddot{p}_1 \\ \ddot{p}_2 \end{Bmatrix} + \begin{bmatrix} 2K & -K \\ -K & K \end{bmatrix} \begin{bmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{bmatrix} \begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{bmatrix}^T \begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix} \begin{bmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{bmatrix} \begin{Bmatrix} \ddot{p}_1 \\ \ddot{p}_2 \end{Bmatrix} + \begin{bmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{bmatrix}^T \begin{bmatrix} 2K & -K \\ -K & K \end{bmatrix} \begin{bmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{bmatrix} \begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$[\boldsymbol{\Psi}]^T[\mathbf{M}][\boldsymbol{\Psi}]\{\ddot{\mathbf{p}}\} + [\boldsymbol{\Psi}]^T[\mathbf{K}][\boldsymbol{\Psi}]\{\mathbf{p}\} = \{\mathbf{0}\}$$

The equations of motion have been converted to modal equations of motion. Are these equations easier to solve?

$$[\mathbf{M}_r]\{\ddot{\mathbf{p}}\} + [\mathbf{K}_r]\{\mathbf{p}\} = \{\mathbf{0}\}$$

Orthogonality of the Modes

The equations are easier to solve provided the modal vectors are orthogonal with respect to the system matrices:

$$[\mathbf{K}]\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \lambda[\mathbf{M}]\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}$$

$$[\mathbf{K}]\{\psi_r\} = \lambda_r[\mathbf{M}]\{\psi_r\}$$

$$[\mathbf{K}]\{\psi_s\} = \lambda_s[\mathbf{M}]\{\psi_s\}$$

$$\{\psi_s^T\}[\mathbf{K}]\{\psi_r\} = \lambda_r\{\psi_s^T\}[\mathbf{M}]\{\psi_r\}$$

$$\{\psi_r^T\}[\mathbf{K}]\{\psi_s\} = \lambda_s\{\psi_r^T\}[\mathbf{M}]\{\psi_s\}$$

$$0 = (\lambda_r - \lambda_s)\{\psi_r^T\}[\mathbf{M}]\{\psi_s\} \quad \text{for } r \neq s$$

This final equation indicates that two different modes *r* and *s* are orthogonal with respect to the mass (and stiffness) matrix.

Uncoupled Equations

Given this proper of mass and stiffness orthogonality of the modal vectors, the modal equations can be written as:

$$\left[\mathbf{M}_r\right]\left\{\ddot{\mathbf{p}}\right\} + \left[\mathbf{K}_r\right]\left\{\mathbf{p}\right\} = \left\{\mathbf{0}\right\} \Rightarrow M_r \ddot{p}_r + K_r p_r = 0 \quad \text{for} \quad r = 1,2$$

$$\text{where} \quad \left\{\psi_r^T\right\}\left[\mathbf{M}\right]\left\{\psi_r\right\} = M_r \quad \text{and} \quad \left\{\psi_r^T\right\}\left[\mathbf{K}\right]\left\{\psi_r\right\} = K_r$$

Therefore, the coordinate transformation $\left\{\mathbf{x}(t)\right\} = \left[\Psi\right]\left\{\mathbf{p}(t)\right\}$ **decouples the coupled set of equations of motion so they can be solved using the single degree of freedom approach discussed earlier.**

Let's prove this to ourselves using the two DOF example.

Two DOF System Example

Consider the case when we have no damping:

$$\begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 2K & -K \\ -K & K \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Assume that *M*=1 kg and *K*=1 N/m, then MATLAB can be used to solve for the undamped modal frequencies / vectors:

$$\begin{Bmatrix} 0.618 \\ 1.000 \end{Bmatrix}, 0.618 rad/s; \quad \begin{Bmatrix} 1.000 \\ -0.618 \end{Bmatrix}, 1.618 rad/s$$

Now assume a solution of the form $\{\mathbf{x}(t)\} = [\Psi]\{\mathbf{p}(t)\}$ **and substitute into the equations of motion to check for orthogonality of the modal vectors:**

Two DOF System Example

Consider the case when we have no damping:

$$\begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix}\begin{bmatrix} 0.618 & 1.000 \\ 1.000 & -0.618 \end{bmatrix}\begin{Bmatrix} \ddot{p}_1 \\ \ddot{p}_2 \end{Bmatrix} + \begin{bmatrix} 2K & -K \\ -K & K \end{bmatrix}\begin{bmatrix} 0.618 & 1.000 \\ 1.000 & -0.618 \end{bmatrix}\begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} 0.618 & 1.000 \\ 1.000 & -0.618 \end{bmatrix}^T \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0.618 & 1.000 \\ 1.000 & -0.618 \end{bmatrix}\begin{Bmatrix} \ddot{p}_1 \\ \ddot{p}_2 \end{Bmatrix}$$

$$+ \begin{bmatrix} 0.618 & 1.000 \\ 1.000 & -0.618 \end{bmatrix}^T \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 0.618 & 1.000 \\ 1.000 & -0.618 \end{bmatrix}\begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix} = \begin{bmatrix} 0.618 & 1.000 \\ 1.000 & -0.618 \end{bmatrix}^T \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} 1.3819 & 0 \\ 0 & 1.3819 \end{bmatrix}\begin{Bmatrix} \ddot{p}_1 \\ \ddot{p}_2 \end{Bmatrix} + \begin{bmatrix} 0.5278 & 0 \\ 0 & 3.6179 \end{bmatrix}\begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

The transformation did in fact decouple the two equations of motion, so these two equations can now be solved!

Modal Properties

Note that the ratios of the modal stiffness, K_r, coefficients to the modal mass coefficients, M_r, are the undamped natural frequencies:

$$\rightarrow \quad \sqrt{\frac{.5278}{1.3819}} = 0.618 rad/s = \omega_{n1}$$

$$\rightarrow \quad \sqrt{\frac{3.6179}{1.3819}} = 1.618 rad/s = \omega_{n2}$$

When we scale the modal vectors differently (unity modal mass, unity vector length, etc.), the modal mass/stiffness coefficients are different but the ratio never changes.

Proportional Viscous Damping

If the viscous damping matrix can be written as a linear combination of the mass and stiffness matrices,

$$\rightarrow \mathbf{C} = \alpha\mathbf{M} + \beta\mathbf{K}$$

then the damping is said to be proportional viscous damping.

For this type of damping, the same modal coordinate transformation procedure leads to:

$$\rightarrow \mathbf{M}_r\ddot{\mathbf{p}} + \mathbf{C}_r\dot{\mathbf{p}} + \mathbf{K}_r\mathbf{p} = \mathbf{0} \Rightarrow M_r\ddot{p}_r + C_r\dot{p}_r + K_r p_r = 0 \text{ for } r = 1,2$$

For non-proportional damping, we use state variable method.