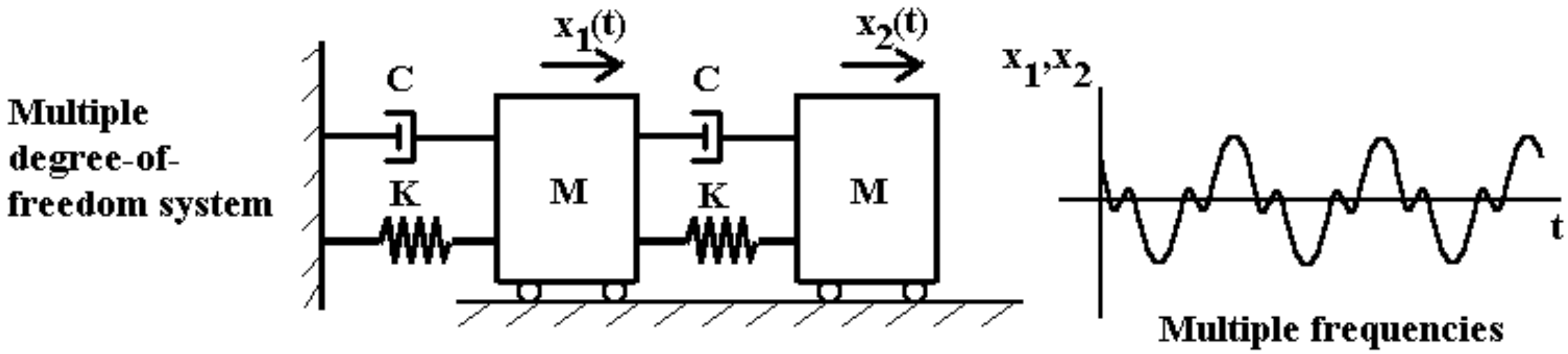
ME 563
Mechanical Vibrations
Lecture #12

Multiple Degree of Freedom

Free Response + MATLAB

Free Response



We can solve for the homogeneous solution to a coupled set of equations in a multiple degree of freedom linear system by:

- Identifying the initial conditions on all the states
- Assuming a solution of the form $\{x(t)\}=\{A\}e^{st}$

What does this last assumption imply about the response?

Two DOF System

Consider the two degree of freedom system of equations:

$$\begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 2C & -C \\ -C & C \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} 2K & -K \\ -K & K \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

If we make a solution of the form, $\{x\} = \{A\} e^{st}$, as we did for the single DOF case, we obtain:

$$\begin{bmatrix} Ms^2 + 2Cs + 2K & -Cs - K \\ -Cs - K & Ms^2 + Cs + K \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} e^{st} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Non-trivial solutions satisfy:

$$\left(Ms^2 + 2Cs + 2K\right) \cdot \left(Ms^2 + Cs + K\right) - \left(Cs + K\right)^2 = 0$$

Characteristic Equation

There are four solutions that satisfy the *characteristic equation* and these solutions are expressed as follows when the modal frequencies are underdamped:

→ $s_1=\sigma_1+j\omega_1$, $s_2=\sigma_1-j\omega_1$, $s_3=\sigma_2+j\omega_2$, and $s_4=\sigma_2-j\omega_2$

Real part: decay rate **Imaginary part: oscillation frequency**

The solution to the homogeneous equation is then written as follows (for the first two complex conjugate roots):

→ $$\{x(t)\}=\{A_1\}e^{s_1t}+\{A_2\}e^{s_2t}+\{A_3\}e^{s_3t}+\{A_4\}e^{s_4t}$$

Free Response Form

The free response is usually written in the following form for a multiple degree of freedom system:

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} X_{11} \\ X_{21} \end{Bmatrix} e^{s_1 t} + \begin{Bmatrix} X^*_{11} \\ X^*_{21} \end{Bmatrix} e^{s_2 t} + \begin{Bmatrix} X_{12} \\ X_{22} \end{Bmatrix} e^{s_3 t} + \begin{Bmatrix} X^*_{12} \\ X^*_{22} \end{Bmatrix} e^{s_4 t}$$

$$= X_1 \begin{Bmatrix} \psi_{11} \\ \psi_{21} \end{Bmatrix} e^{\sigma_1 t} \cos(\omega_1 t + \phi_1) + X_2 \begin{Bmatrix} \psi_{12} \\ \psi_{22} \end{Bmatrix} e^{\sigma_2 t} \cos(\omega_2 t + \phi_2)$$

Modal vector (can be scaled)

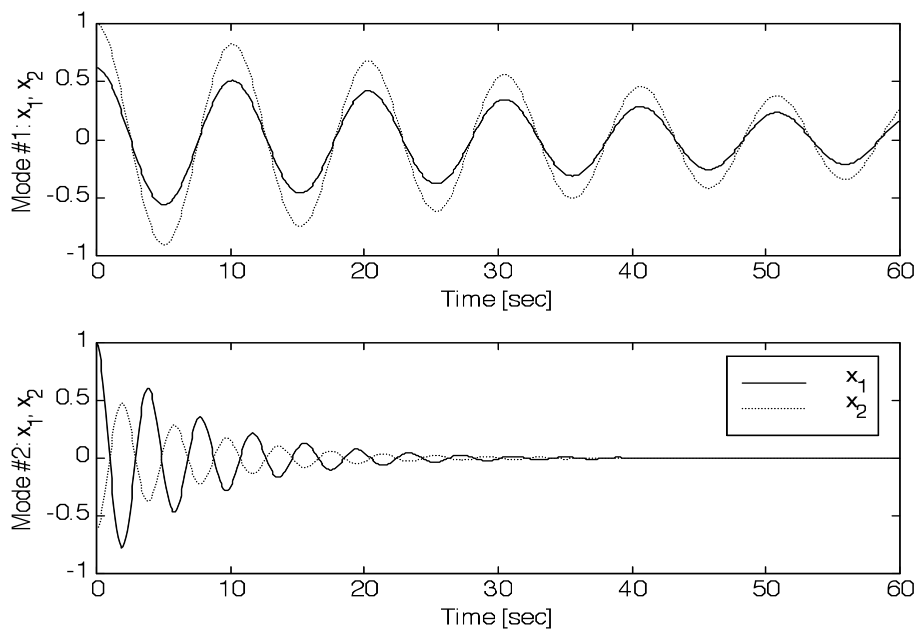
Decaying co-sinusoid (common to both degrees of freedom)

Four constants → Four initial conditions are required.

Example Solution

For *K=1 N/m*, *C=0.1 Ns/m*, and *M=1 kg*, **the solution is given by:**

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = X_1 \begin{Bmatrix} 0.618 \\ 1.000 \end{Bmatrix} e^{-0.019t} \cos(0.618t + \phi_1) + X_2 \begin{Bmatrix} 1.000 \\ -0.618 \end{Bmatrix} e^{-0.131t} \cos(1.613t + \phi_2)$$



MATLAB Solution

To obtain solutions for the free response in MATLAB, the procedure explained before for a single DOF system is used:

$$\frac{d}{dt}\begin{Bmatrix} x_1 \\ x_2 \\ \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix}_{4\times1} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{2K}{M} & \frac{K}{M} & -\frac{2C}{M} & \frac{C}{M} \\ \frac{K}{M} & \frac{K}{M} & \frac{C}{M} & -\frac{C}{M} \end{bmatrix}_{4\times4} \begin{Bmatrix} x_1 \\ x_2 \\ \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix}_{4\times1} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}_{4\times2} \begin{Bmatrix} f_1(t) \\ f_2(t) \end{Bmatrix}$$

Or

$$\frac{d}{dt}\begin{Bmatrix} x_1 \\ x_2 \\ \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix}_{4\times1} = \begin{bmatrix} [0] & [I] \\ -[M]^{-1}[K] & -[M]^{-1}[C] \end{bmatrix}_{4\times4} \begin{Bmatrix} x_1 \\ x_2 \\ \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix}_{4\times1} + \begin{bmatrix} [0] \\ [M]^{-1}[0] \end{bmatrix}_{4\times2} \begin{Bmatrix} f_1(t) \\ f_2(t) \end{Bmatrix}$$

MATLAB Solution

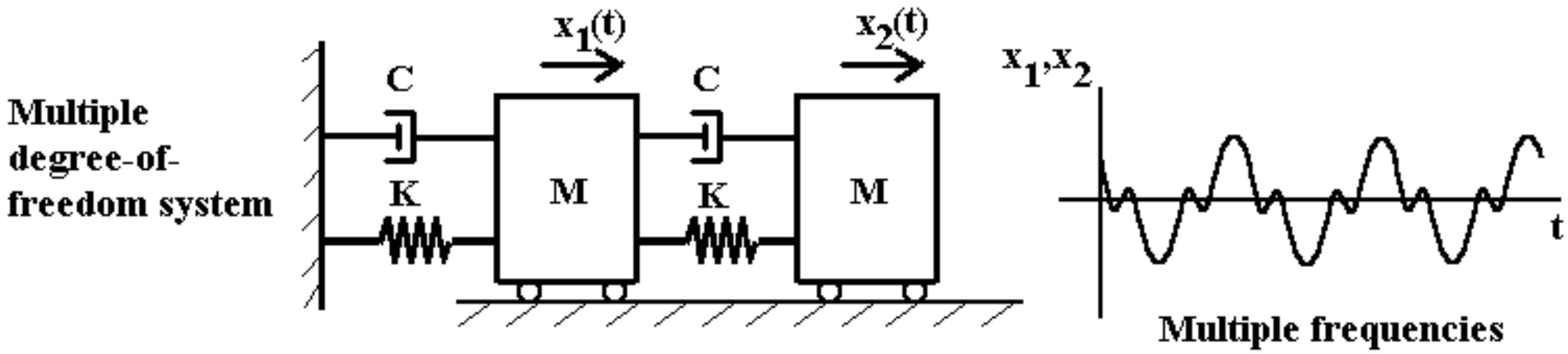
We use the following formulation to define the outputs, *{y}*, if we are only interested in the displacement variables:

→
$$\begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix}_{2\times 1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}_{2\times 4} \begin{Bmatrix} x_1 \\ x_2 \\ \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix}_{4\times 1} + \begin{bmatrix} 0 \end{bmatrix}_{2\times 2} \begin{Bmatrix} f_1(t) \\ f_2(t) \end{Bmatrix}$$

Or:

→
$$\{y\}_{2\times 1} = [C]_{2\times 4}\{q\}_{4\times 1} + [D]_{2\times 2}\{f(t)\}_{2\times 1}$$

Free Response (Eigen-Analysis)



We can also solve the homogeneous equations of motion by:

- Identifying the initial conditions on all the states
- Identifying the modal frequencies, *s*, and vectors, *{X}*, using eigen-analysis.

Do you remember what an eigenvalue problem looks like?

Eigenvalue Problem

Consider the case when we have no damping:

$$\rightarrow \begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 2K & -K \\ -K & K \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

We can write this set of equations of motion as follows:

$$\rightarrow \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix}^{-1} \begin{bmatrix} 2K & -K \\ -K & K \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\rightarrow \begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix}^{-1} \begin{bmatrix} 2K & -K \\ -K & K \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = -\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix}$$

Now assume a solution of the form $\{x(t)\}=\{A\}e^{st}$

Eigenvalue Problem

Consider the case when we have no damping:

$$\begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix}^{-1} \begin{bmatrix} 2K & -K \\ -K & K \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = -s^2 \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}$$

$$[A] \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \lambda \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}$$

$$\left([A] - \lambda [I]\right) \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \{0\}$$

This last equation is the standard eigenvalue problem. Matlab solves this equation with the command "eig(A)":

```
[v,d]= eig(A);
```

Matlab `eig`

How do we interpret what Matlab gives us with this command?

`d` is a diagonal matrix of $-s_{1,2}^2$
`v` is a matrix of modal vectors

→ **d =**

2.6180	0
0	0.3820

→ **v =**

-0.8507	-0.5257
0.5257	-0.8507

How do we find the modal frequencies and vectors?

Matlab `eig`

First, we take the negative square root of the diagonal entries of the `d` matrix providing the modal frequencies in rad/s:

→ √-d =

0+1.6180i	0
0	0+0.6181i

Second, we can scale the columns of v to produce easy to interpret modal vectors:

→ v =

1.0000	0.6180
-0.6180	1.0000

Matlab Form of Response

The form of the free response can then be written in the form:

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = X_1 \begin{Bmatrix} 0.618 \\ 1.000 \end{Bmatrix} e^{0t} \cos(0.618t + \phi_1) + X_2 \begin{Bmatrix} 1.000 \\ -0.618 \end{Bmatrix} e^{0t} \cos(1.618t + \phi_2)$$

In summary, the eigenvalue problem for an undamped system is given by:

Where does the damping fit?

$$\begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix}^{-1} \begin{bmatrix} 2K & -K \\ -K & K \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = -s^2 \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}$$

Damped Eigenvalue Problem

To obtain solutions for the free response in a damped system, the state variable form of the equations of motion are used:

$$\frac{d}{dt}\begin{Bmatrix} x_1 \\ x_2 \\ \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix}_{4\times 1} = \begin{bmatrix} [0] & [I] \\ -[M]^{-1}[K] & -[M]^{-1}[C] \end{bmatrix}_{4\times 4} \begin{Bmatrix} x_1 \\ x_2 \\ \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix}_{4\times 1} + \begin{bmatrix} [0] \\ [M]^{-1}[0] \end{bmatrix}_{4\times 2} \begin{Bmatrix} f_1(t) \\ f_2(t) \end{Bmatrix}$$

and then the eigenvalues and eigenvectors of the state matrix are calculated using `eig`. This approach works because the assumed solution *{q}*e^{st} is also used for the 1st order system:

```
[v,d]= eig(A);
```

Matlab `eig`

How do we interpret what Matlab gives us with this command in the first order form of the state variable model?

`d` **is a diagonal matrix of modal frequencies** $s_{1,2}$ **and** $s_{3,4}$

`v(1:2,1:4)` **is a matrix of modal vectors** $\{X\}_1$ **and** $\{X^*\}_1$

$\{X\}_2$ **and** $\{X^*\}_2$

→ **diag(d) =**

-0.1309 + 1.6127i $s_{3,4}$
-0.1309 - 1.6127i
-0.0191 + 0.6177i $s_{1,2}$
-0.0191 - 0.6177i

Matlab `eig`

Eigenvectors are not the modal vectors of the 2nd order system – eigenvectors involve both position and velocity states. We look to the position states to obtain the scaled modal vectors.

→ v = [

-0.0362 - 0.4457i   -0.0362 + 0.4457i → $\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}_2$
0.0224 + 0.2755i   0.0224 - 0.2755i
0.7236   0.7236
-0.4472 + 0.0000i   -0.4472 - 0.0000i

-0.4472 + 0.0000i   -0.4472 - 0.0000i → $\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}_1$
-0.7236   -0.7236
0.0085 - 0.2763i   0.0085 + 0.2763i
0.0138 - 0.4470i   0.0138 + 0.4470i

Damping in MDOF Systems

Note that for the previous example, the damping matrix *[C]* corresponded to a proportionally viscously damped system:

$$[C] = \alpha[K] + \beta[M]$$

$$\alpha, \beta \quad \text{Real constants}$$

For this type of damping, note that the modal vectors were entirely <u>real</u>, i.e., the phase between DOFs was either 0 or 180^o.

$$\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}_1 = \begin{Bmatrix} 0.618 \\ 1.000 \end{Bmatrix}, \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}_2 = \begin{Bmatrix} 1.000 \\ -0.618 \end{Bmatrix}$$

Damping in MDOF Systems

What if the system is not proportionally damped? The modal vectors in this case are <u>complex</u> but the response is not:

→ $$[C] \neq \alpha[K] + \beta[M] = \begin{bmatrix} .2 & -.2 \\ -.2 & .2 \end{bmatrix}$$

For this type of damping, note that the modal vectors are <u>complex</u>, so the phase between the DOFs is not 0 or 180°.

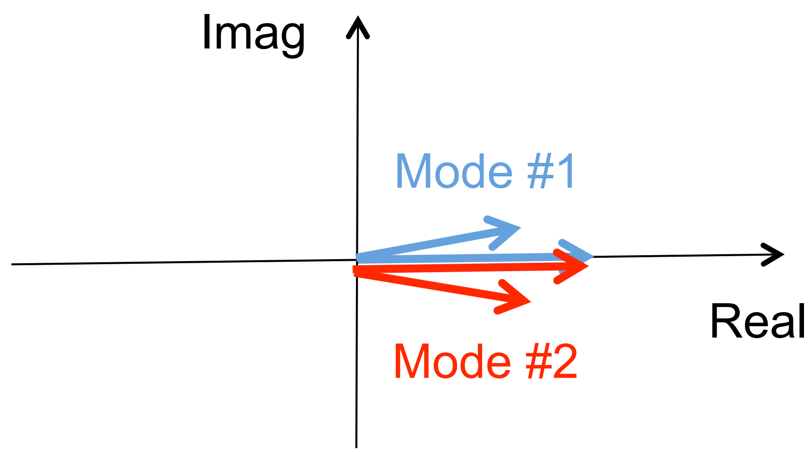
→

$$\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}_1 = \begin{Bmatrix} 0.6203 + 0.0341i \\ 1.000 \end{Bmatrix}, \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}_2 = \begin{Bmatrix} 1.000 \\ -0.6272 - 0.0893i \end{Bmatrix}$$

What does this mean?

Damping in MDOF Systems

What if the system is not proportionally damped? The modal vectors in this case are <u>complex</u> but the response is not:



$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = X_1 \begin{Bmatrix} 0.621\cos\left(0.618t + 2.1^o + \phi_1\right) \\ 1.000\cos\left(0.618t + \phi_1\right) \end{Bmatrix} e^{-0.019t} + X_2 \begin{Bmatrix} 1.000\cos\left(1.613t + \phi_2\right) \\ 0.634\cos\left(1.613t - 171^o + \phi_2\right) \end{Bmatrix} e^{-0.131t}$$

2nd order EOMS cannot be uncoupled using real modal vectors!