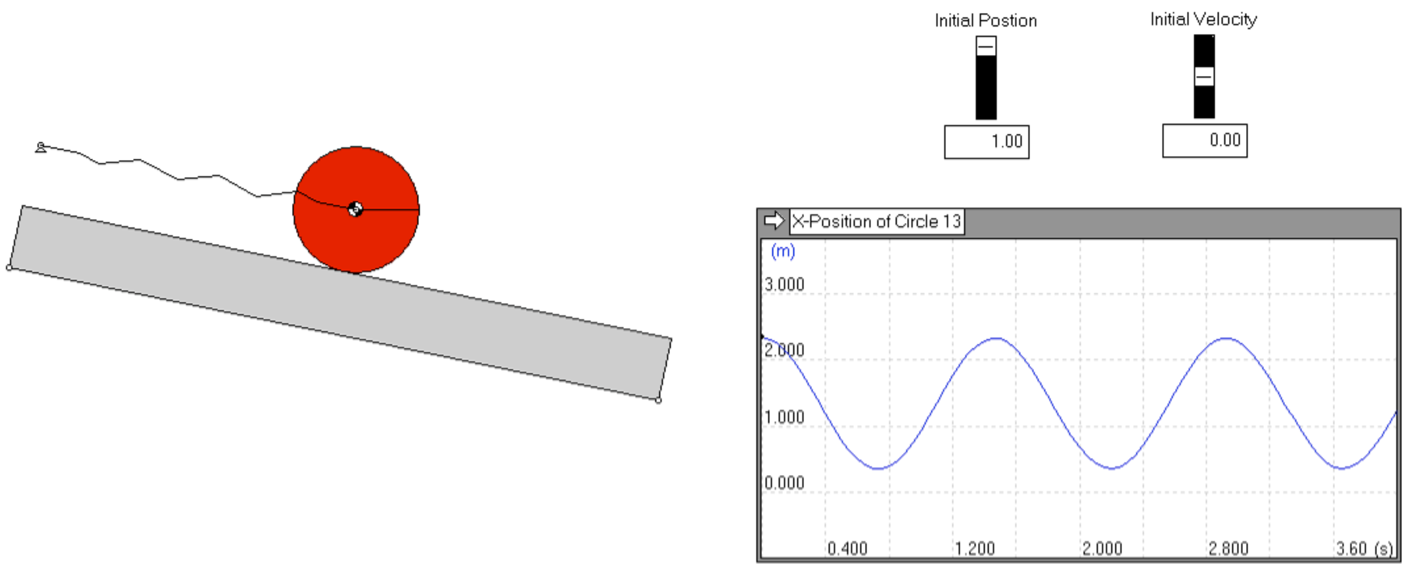
ME 563
Mechanical Vibrations
Lecture #10

Single Degree of Freedom

Free Response

Free Response



When solving the homogeneous equation of motion (forcing function = 0), we are finding the free response. One way to solve for the free response is as follows:

- Identify the initial conditions on all the states

- Assume a solution of the form $x(t)=Ae^{st}$

Disc on An Incline

Consider the disc on an incline with no forcing function:

$$\rightarrow \quad \left(M + \frac{I_{CM}}{a^2}\right)\ddot{x}_d + C\dot{x}_d + Kx_d = 0$$

Recall x_d is called the dynamic displacement.

If we make a solution of the form, $x_d = Ae^{st}$, we obtain:

$$\rightarrow \quad \left(M + \frac{I_{CM}}{a^2}\right)s^2 Ae^{st} + CsAe^{st} + KAe^{st} = 0$$

$$\rightarrow \quad \left[\left(M + \frac{I_{CM}}{a^2}\right)s^2 + Cs + K\right]Ae^{st} = 0$$

Characteristic Equation

The only solutions that satisfy the equation of motion that are not trivial (A=0) must also satisfy the *characteristic equation*:

$$\rightarrow \quad \left(M + \frac{I_{CM}}{a^2}\right)s^2 + Cs + K = 0$$

The solution to the homogeneous equation are then written as follows:

Roots, poles, modal frequencies

$$\rightarrow \quad x_d(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \quad \text{where} \quad s_{1,2} = \frac{-C}{2\left(M + \frac{I_{CM}}{a^2}\right)} \pm \frac{\sqrt{C^2 - 4\left(M + \frac{I_{CM}}{a^2}\right)K}}{2\left(M + \frac{I_{CM}}{a^2}\right)}$$

Because the equation is linear, the solutions superimpose…

Initial Conditions

It is important to note that the modal frequencies of the system are only a function of the mass, damping, and stiffness parameters and not a function of the initial conditions or forcing function.

The constants A_1 and A_2 are found by applying the initial conditions to the solution that was obtained:

$$x_d(0) = A_1 + A_2$$
$$\dot{x}_d(0) = s_1 A_1 + s_2 A_2$$

Or

$$\begin{bmatrix} 1 & 1 \\ s_1 & s_2 \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = \begin{Bmatrix} x_d(0) \\ \dot{x}_d(0) \end{Bmatrix}$$

Alternative Solution Method

Another method to solve for the free response is to transform the equation of motion into the complex frequency domain using the Laplace transform:

$$\left[\left(M+\frac{I_{CM}}{a^2}\right)s^2-\left(M+\frac{I_{CM}}{a^2}\right)sx_d(0)-\left(M+\frac{I_{CM}}{a^2}\right)\dot{x}_d(0)\right]X_d(s)+\left[Cs-Cx_d(0)\right]X_d(s)+KX_d(s)=0$$

$$\left[\left(M+\frac{I_{CM}}{a^2}\right)s^2+Cs+K\right]X_d(s)=\left(M+\frac{I_{CM}}{a^2}\right)sx_d(0)+Cx_d(0)+\left(M+\frac{I_{CM}}{a^2}\right)\dot{x}_d(0)$$

$$X_d(s)=\frac{\left(M+\frac{I_{CM}}{a^2}\right)s+C}{\left(M+\frac{I_{CM}}{a^2}\right)s^2+Cs+K}x_d(0)+\frac{\left(M+\frac{I_{CM}}{a^2}\right)}{\left(M+\frac{I_{CM}}{a^2}\right)s^2+Cs+K}\dot{x}_d(0)$$

Note the initial conditions are embedded in the process.

The Roots of the Response

The roots of the characteristic equation determine the nature of the free response. This relationship between the roots and the free response can be represented graphically by plotting the real and imaginary parts of the roots in the complex plane.

First, the roots are expressed as follows to show our preference for complexity (because we only expect free oscillations when the roots are complex):

Terminology…

$$s_{1,2} = -\zeta\omega_n \pm j\omega_d = \sigma \pm j\omega_d$$

$$\sigma = \frac{-C}{2M_{eff}}, \zeta = \frac{C}{2\sqrt{KM_{eff}}}, \omega_n = \sqrt{\frac{K}{M_{eff}}}, \omega_d = \sqrt{1-\zeta^2}\,\omega_n$$

Visualizing the Free Response

The roots are then plugged into the solution:

$$\rightarrow \quad x_d(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

We then learn that there are three types of (stable) roots:

Overdamped: No free oscillations can take place

$$\rightarrow \quad s_{1,2} = -R_1, -R_2 \Rightarrow x_d(t) = A_1 e^{-R_1 t} + A_2 e^{-R_2 t}$$

Critically damped: No free oscillations can take place

$$\rightarrow \quad s_{1,2} = -R_1 \Rightarrow x_d(t) = A_1 e^{-R_1 t} + A_2 t e^{-R_1 t}$$

Underdamped: Free oscillations can take place

$$\rightarrow \quad s_{1,2} = \sigma \pm j\omega_d \Rightarrow x_d(t) = e^{\sigma t}\left(A_1 e^{j\omega_d t} + A_2 e^{-j\omega_d t}\right)$$

Real Underdamped Response

For an underdamped free response, the solution can be written in a different way if we consider real responses only:

$$x_d(t) = e^{\sigma t}\left(A_1 e^{j\omega_d t} + A_2 e^{-j\omega_d t}\right)$$

$$= e^{\sigma t}\left(A_1\left(\cos\omega_d t + j\sin\omega_d t\right) + A_2\left(\cos\omega_d t - j\sin\omega_d t\right)\right)$$

$$= e^{\sigma t}\left(\left(A_{1r} + jA_{1i}\right)\left(\cos\omega_d t + j\sin\omega_d t\right) + \left(A_{2r} + jA_{2i}\right)\left(\cos\omega_d t - j\sin\omega_d t\right)\right)$$

$$= e^{\sigma t}\begin{pmatrix}\left(A_{1r} + A_{2r}\right)\cos\omega_d t + \left(A_{2i} - A_{1i}\right)\sin\omega_d t \\ +j\left(A_{1i} + A_{2i}\right)\cos\omega_d t + j\left(A_{1r} - A_{2r}\right)\sin\omega_d t\end{pmatrix}$$

What is the condition required for a real response?

$$\begin{matrix} A_{1i} + A_{2i} = 0 \\ A_{1r} - A_{2r} = 0 \end{matrix} \Rightarrow A_1 = a + bj = A_2^*$$

Simplified Form of Response

Therefore, it is more straightforward to write the free response in its real form:

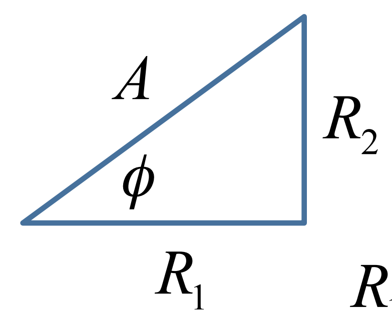
$$x_d(t) = e^{\sigma t}\left(\left(A_{1r} + A_{2r}\right)\cos\omega_d t + \left(A_{2i} - A_{1i}\right)\sin\omega_d t\right)$$

$$= e^{\sigma t}\left(2a\cos\omega_d t - 2b\sin\omega_d t\right)$$

$$= e^{\sigma t}\left(R_1\cos\omega_d t - R_2\sin\omega_d t\right)$$

$$= Ae^{\sigma t}\cos\left(\omega_d t + \phi\right)$$

$$A = \sqrt{R_1^2 + R_2^2}, \phi = a\tan\frac{R_2}{R_1}$$



$$R_1 = A\cos\phi$$

$$R_2 = A\sin\phi$$

$$Ae^{\sigma t}\cos\left(\omega_d t + \phi\right)$$

Simplified Form of Response

After applying the initial conditions, the free response is obtained:

$$\rightarrow Ae^{\sigma t}\cos\left(\omega_d t + \phi\right)$$

where,

$$\phi = \tan^{-1}\frac{-\dot{x}_d(0) + \sigma x_d(0)}{\omega_d}$$

$$A = \sqrt{\left(\frac{-\dot{x}_d(0) + \sigma x_d(0)}{\omega_d}\right)^2 + \left(x_d(0)\right)^2}$$

Notes about Free Response

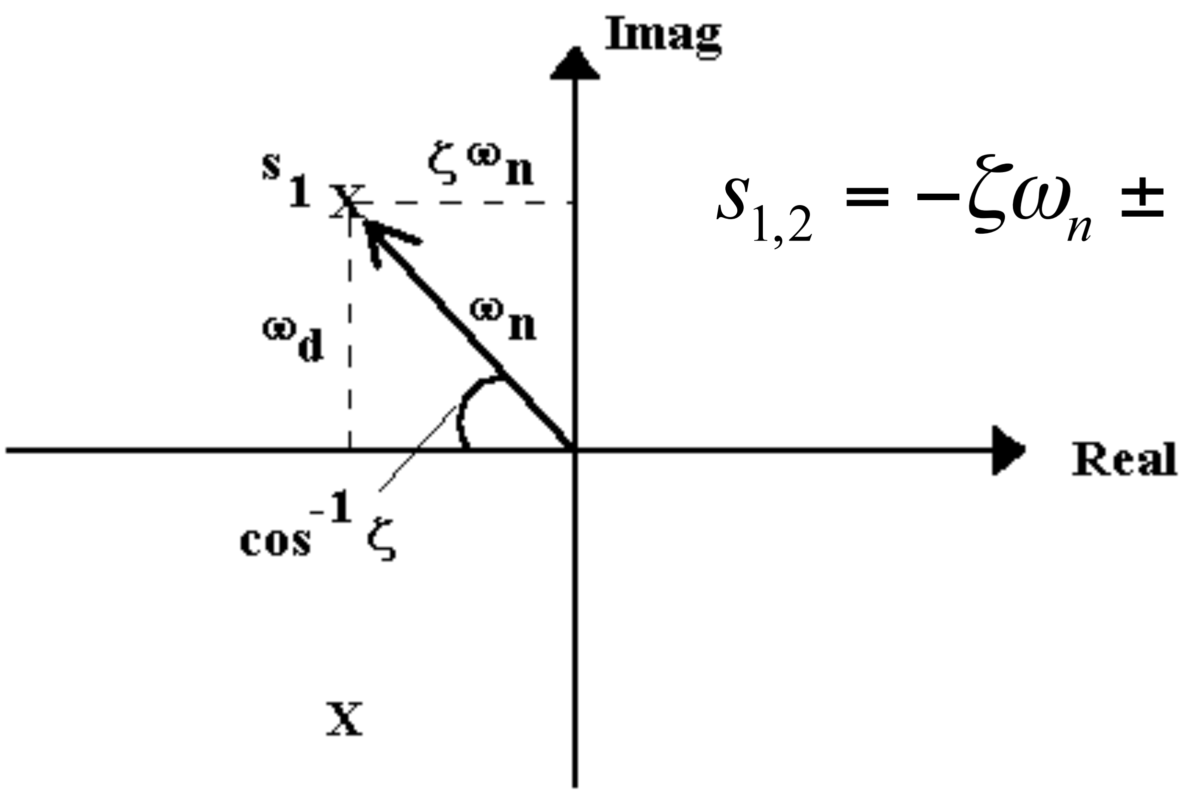
There are several important things to note about the free response:

$$s_{1,2} = \sigma \pm j\omega_d \Rightarrow Ae^{\sigma t}\cos(\omega_d t + \phi)$$

-Real part of the root determines the decay (damping factor)
-Imaginary part of the root determines oscillation frequency (damped natural frequency)
-When the damping coefficient is positive, the real part of the root is negative and the free response is stable.
-When the damping coefficient is zero, the roots are imaginary and the free response is marginally stable (oscillates forever).

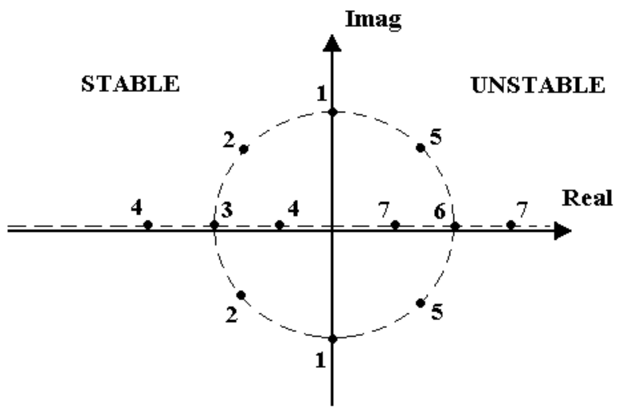
Roots in Complex Plane

The underdamped roots can now be plotted in the complex plane using the following geometric rule that makes use of the modal parameters (damping ratio, natural frequencies):



$$s_{1,2} = -\zeta\omega_n \pm j\omega_d = \sigma \pm j\omega_d$$

Visualizing the Free Response

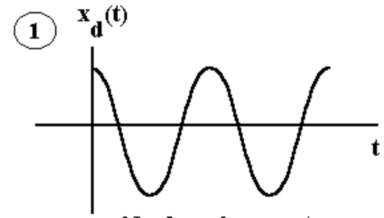


(1) Undamped (oscillations)
- Roots imaginary

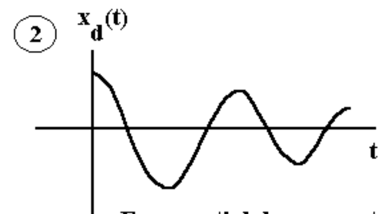
(2) Underdamped (oscillations with decay)
- Roots complex conjugate

(3) Criticallly damped (no oscillations; repeated)
- Roots real and equal

(4) Overdamped (no oscillations; real roots)
- Roots real and unequal

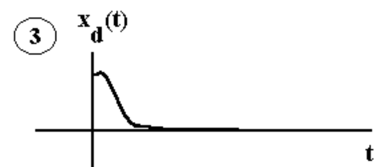


No damping; roots on imaginary axis



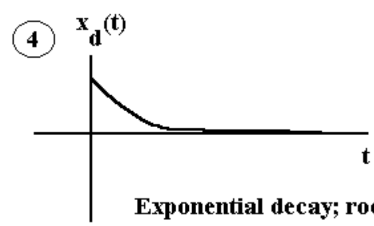
Exponential decay; roots in left-half plane

(5) Exponentially diverging unstable version of (2)



Exponential decay with time; repeated roots on left real axis

(6) Exponentially diverging unstable version of (3)



Exponential decay; roots on left real axis

(7) Exponentially diverging unstable version of (4)

Response Characteristics

How long does it take for the response to decay?

$$e^{\sigma t} = e^{-t/\tau_s} \Rightarrow \boldsymbol{\tau}_s \rightarrow 63\%, 4\boldsymbol{\tau}_s \rightarrow 98\%$$

What system parameters determine the decay rate?

$$\sigma = -\frac{C}{2M} = -\zeta\omega_n$$

What is the period of oscillation?

$$T = \frac{2\pi}{\omega_d} = \frac{2\pi}{\sqrt{1-\zeta^2}\,\omega_n}$$

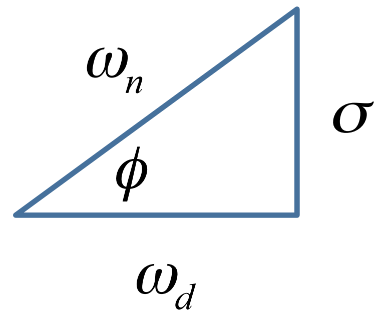
Example #1

For initial position 1m and initial velocity 0m/s, find $x_d(t)$.

→ $$x_d(t) = Ae^{\sigma t}\cos(\omega_d t + \phi)$$

→ $$x_d(0) = 1 = A\cos(\phi)$$

→ $$\dot{x}_d(0) = A\sigma\cos(\phi) - A\omega_d\sin(\phi) = 0$$



→ $$\frac{\sin(\phi)}{\cos(\phi)} = \frac{\sigma}{\omega_d} \Rightarrow \phi = a\tan\frac{\sigma}{\omega_d}$$

→ $$A = 1/\cos\phi \Rightarrow A = \frac{\omega_n}{\omega_d}$$

Second option…

$$\phi = a\tan\frac{\sigma x_d(0)}{\omega_d}$$

$$A = \sqrt{\left(\frac{\sigma x_d(0)}{\omega_d}\right)^2 + (x_d(0))^2}$$

Example #2

For initial position 0m and initial velocity 1/M m/s, find $x_d(t)$.

$$\rightarrow x_d(t) = Ae^{\sigma t}\cos(\omega_d t + \phi)$$

$$\rightarrow x_d(0) = 0 = A\cos(\phi)$$

$$\rightarrow \dot{x}_d(0) = A\sigma\cos(\phi) - A\omega_d\sin(\phi) = 1/M$$

Impulse response

$$x_d(t) = \frac{e^{\sigma t}}{M\omega_d}\sin(\omega_d t)$$

$$\rightarrow \cos\phi = 0 \Rightarrow \phi = n\frac{\pi}{2}, n = \pm 1,3,5,\ldots$$

$$\rightarrow A = \frac{1/M}{\sigma\cos(\phi) - \omega_d\sin(\phi)} \Rightarrow A = \frac{-1}{M\omega_d}$$