**PROBLEM 1:**
Given the mass matrix and two undamped natural frequencies for a general two degree-of-freedom system with a symmetric stiffness matrix,

\[
[M] = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad \omega_{n1} = 0 \text{ rad/s}, \omega_{n2} = \sqrt{3} \text{ rad/s}
\]

find the stiffness matrix, modal mass and stiffness for each mode of vibration, and the modal vectors (normalize the modal vectors so that the largest coefficient in each vector is of unity magnitude).

First, note that the system has a rigid body mode, which we assume is described by the modal vector \([1 \ 1]^T\) (i.e., semi-definite system). Also, recall that modal orthogonality can be expressed in the following way for undamped systems like this one with distinct modal frequencies:

\[
\begin{align*}
\psi_1^T[M]\psi_1 &= M_1 \\
\psi_2^T[M]\psi_2 &= M_2 \\
\psi_1^T[K]\psi_1 &= K_1 \\
\psi_2^T[K]\psi_2 &= K_2 \\
\psi_1^T[M]\psi_2 &= 0 \\
\psi_2^T[K]\psi_1 &= 0
\end{align*}
\]

where \(\frac{K_1}{M_1} = 0 \text{ rad/s} \) and \(\frac{K_2}{M_2} = 3 \text{ rad/s} \)

\[
\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad \omega_{n1} = 0 \text{ rad/s}, \omega_{n2} = \sqrt{3} \text{ rad/s}
\]

The equation \(K_1/M_1=0\) implies that the modal stiffness for mode 1 is zero, so with the known form of the modal vector, we can find the stiffness relationship:

\[
\psi_1^T[K_1 \quad K_{12} \quad K_{12} \quad K_{22}]
\psi_1 = K_1 = \psi_1^T\begin{bmatrix} K_{11} + K_{12} \\ K_{12} + K_{22} \end{bmatrix} = K_{11} + 2K_{12} + K_{22} = 0 \quad (1)
\]

But we also know that for semi-definite systems, the stiffness matrix is singular:

\[
K_{11}K_{22} - K_{12}^2 = 0 \quad (2)
\]

With three unknowns and two equations (1) and (2), a third equation is needed that relates the stiffness parameters. This equation is found by moving to the second mode orthogonality expression:

\[
\psi_1^T[K_1 \quad K_{12} \quad K_{12} \quad K_{22}]
\psi_2 = 0 = \psi_1^T\begin{bmatrix} K_{11}\psi_{12} + K_{12}\psi_{22} \\ K_{12}\psi_{12} + K_{22}\psi_{22} \end{bmatrix} = (K_{11} + K_{12})\psi_{12} + (K_{12} + K_{22})\psi_{22} = 0 \quad (3)
\]

However, this equation contains two more unknowns, the second modal vector coefficients, that must now be associated with two additional equations to solve the set. These two equations are also found using mass orthogonality relationships:
These equations can now be solved simultaneously to obtain the five desired unknowns:

From Eq. (4), $\psi_{22} = -\frac{1}{2} \psi_{12}$. Substituting this result into Eq. (3) yields $2K_{11} + K_{12} - K_{22} = 0$, which when added to Eq. (1) yields $3K_{11} + 3K_{12} = 0$ or $K_{12} = -K_{11}$. Then this result can be back-substituted into Eq. (1) to determine that $K_{22} = K_{11}$. Thus, the stiffness matrix is of the form

$$
K_{11} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.
$$

When these results are all substituted into Eq. (2), the following conclusion is drawn: $K_{11}^2 - K_{11}^2 = 0$, which is an identity and provides no further useful information. From Eq. (5),

$$
3 = \frac{K_{11} \psi_{12}^2 + K_{11} \psi_{12}^2 + K_{11} \psi_{12}^2}{\psi_{12}^2 + \frac{1}{2} \psi_{12}^2} = \frac{9}{4} \frac{K_{11}}{3} = \frac{3}{2} K_{11} \Rightarrow K_{11} = 2.
$$

The stiffness matrix is therefore given by

$$
\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}.
$$

It was already determined in the previous development that the modal stiffness for mode 1 is zero and that the modal vectors are defined according to the ratios,

$$
\left\{ \psi_{11} \right\} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \left\{ \psi_{12} \right\} = \begin{bmatrix} 1 \\ -0.5 \end{bmatrix}.
$$

Remember that modal vectors are only defined to within a scale factor (they are under-determined). With these particular modal vector scaling selections, the full set of modal mass and modal stiffness values can be summarized as follows:
\[ M_1 : \psi_1^T \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \psi_1 = M_1 = \psi_1^T \begin{bmatrix} \psi_{11} \\ 2\psi_{21} \end{bmatrix} = 1 + 2 = 3 \]

\[ K_1 : \quad K_1 = 0 \]

\[ M_2 : \psi_2^T \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \psi_2 = M_2 = \psi_2^T \begin{bmatrix} \psi_{12} \\ 2\psi_{22} \end{bmatrix} = 1 + 2 \left(0.5^2\right) = 1.5 \]

\[ K_2 : \quad K_2 = M_2 \times 3 = 4.5 \]
PROBLEM 2:

Calculate, plot (in MATLAB), and describe the response of the single story building in Figure 1 to the two different blast excitations, \( f(t) \), shown. Assume \( M=100,000 \) kg, \( K=100 \) kN/m, \( C=1000 \) N-s/m, \( t_o=4 \) sec, \( T_o=12 \) sec, and \( F_o=10 \) kN. What happens to the responses as \( t_o \) approaches zero?

The first thing we have to do is describe the excitations above analytically; only then can we use the frequency response function to calculate the response. We use a different approach for each input. On the left, the excitation is a transient excitation – it starts at a certain time and ends a later time. On the right, the excitation is a periodic excitation – it starts at \(-\infty\) and ends at \(+\infty\).

Since we have already studied the forced response of vibrating systems to step (static) inputs, it makes sense to use these results if possible. To that end, note that the transient excitation can be described as shown below:

This means that we can find the total forced response solution by finding the response to each individual excitation separately. We have to be careful though because remember that the total
solution is the sum of the complimentary and particular solutions, so we need to be cautious about how we choose the initial conditions for each component of the forced response.

For the $f_1(t)$ component, we proceed exactly as we did in Chapter 4 of the notes:

$$x_{p1}(t) = \frac{F_o}{K} \left[ 1 - \frac{\omega_n}{\omega_d} e^{\sigma t} \cos \left( \omega_d t - \tan^{-1} \frac{\xi}{\sqrt{1 - \xi^2}} \right) \right]$$

where $u_s(t) = \begin{cases} 0 & \text{for } t < 0 \\ 1 & \text{for } t > 0 \end{cases}$

This gives the total solution to zero initial conditions, which we will assume is the case in the one story building before the blast occurs.

Now we proceed to find the solution to the $f_2(t)$ component. If we do not want to affect the solution above at $t=t_o$, then we should enforce zero initial conditions for $x_2(t)$ as well. This means the solution to the step down ($-F_o$) is the same as the solution above except it is time shifted to the right by $t_o$ seconds:

$$x_{p2}(t) = \frac{F_o}{K} \left[ 1 - \frac{\omega_n}{\omega_d} e^{\sigma(t-t_o)} \cos \left( \omega_d (t-t_o) - \tan^{-1} \frac{\xi}{\sqrt{1 - \xi^2}} \right) \right] u_s(t-t_o) = x_2(t) \quad \text{and} \quad x_2(0) = 0 = \dot{x}_2(0)$$

The total solution is the sum of these solutions:

$$x(t) = x_1(t) + x_2(t)$$

Note that the initial conditions of this solution are zero and the displacement and velocity at time $t_o$ are determined only by the solution for $x_1(t)$ as desired.

The total solution is plotted below. Note that the steady state response is zero as expected, but the system is very lightly damped (0.5 percent of critical damping) so the response rings down for a very long time. Also, note the most important thing: even though the static response of this system to an input force of $F_o$ magnitude is $F_o/K=0.1m=10$ cm, the maximum dynamic response reaches almost 20 cm, which means that if we do not consider the dynamic response when designing the building, we are not likely to withstand such a blast excitation. This is why structures must be designed with dynamics and vibrations in mind...mechanical systems respond very differently to dynamic forces than to static ones.
What happens when $t_o$ approaches zero? From the figure above, it might be intuitively obvious that the delayed negative step response begins to 'cancel out' the initial positive step up so that the resultant total response becomes smaller in amplitude. Furthermore, the initial condition on the displacement remains zero BUT the initial condition on the velocity is effectively changed to something nonzero. We will prove this later when we talk about impulse response functions. Finally note that the pulse turns into an impulse as $t_o$ approaches zero only if we allow $F_o$ to approach infinity. The response for $t_o=0.02$ sec is shown below – it is very close to a damped sinusoid with a small amplitude.
The response to the periodic input must be approached differently. In this case, we will focus only on the steady state response because we know that the excitation repeats itself and that the transient will eventually decay to zero. In order to calculate the response, we need to analytically describe the excitation using Fourier series. We use the technique reviewed in class to obtain the following Fourier series coefficients and then we apply the approach in Figure 4.2. First, we find the Fourier series:

\[
f(t) = \sum_{n=0}^{\infty} \left[ a_n \cos(n\omega_o t) + b_n \sin(n\omega_o t) \right]
\]

where \( a_n = \frac{2}{T_o} \int_{-T_o/2}^{T_o/2} f(t) \cos(n\omega_o t) dt \Rightarrow a_o = \frac{1}{T_o} \int_{0}^{T_o} F_o dt = \frac{F_o T_o}{T_o} \)

\[
= \frac{2}{T_o} \int_{0}^{T_o} F_o \cos(n\omega_o t) dt
\]

\[
= \frac{2F_o}{T_o} \left[ \frac{1}{n\omega_o} \sin(n\omega_o t) \right]_0^{T_o} = \frac{2F_o}{T_o n\omega_o} \sin(n\omega_o T_o) = \frac{F_o}{\pi n} \sin \left( n2\pi \frac{T_o}{T_o} \right)
\]

\( b_n = \frac{2}{T_o} \int_{-T_o/2}^{T_o/2} f(t) \sin(n\omega_o t) dt \)

\[
= \frac{2}{T_o} \int_{0}^{T_o} F_o \sin(n\omega_o t) dt
\]

\[
= \frac{2F_o}{T_o} \left[ -\frac{1}{n\omega_o} \cos(n\omega_o t) \right]_0^{T_o} = -\frac{2F_o}{T_o n\omega_o} \cos(n\omega_o t_o) + \frac{2F_o}{T_o n\omega_o} = -\frac{F_o}{\pi n} \left[ \cos \left( n2\pi \frac{T_o}{T_o} \right) - 1 \right]
\]

The 20 term (n=20) approximation to \( f(t) \) is shown below. Note the ringing due to the so-called Gibbs phenomenon. Even though this approximation looks fairly poor (compare this result with the actual periodic blast), it so happens that these imperfections in our description of the excitation (i.e. Gibbs phenomenon) are unimportant in our forced response analysis because the system ‘filters’ the high frequency ringing out of the response. This is obvious from the ‘mass line’ of the FRF.
We can show this by applying the concept in Figure 4.2 in the class notes. We will use the frequency response function of the SDOF one-story building model to compute the steady state response to the periodic excitation above for various numbers of terms in the Fourier series.

The steady state response (particular solution) is given by:

\[
x_p(t) = \sum_{n=0}^{\infty} \left[ a_n \cos(n\omega_o t + \angle H(n\omega_o)) + b_n \sin(n\omega_o t + \angle H(n\omega_o)) \right]
\]

\[
= \frac{F_o}{T_o} \| H(0) \| + \\
\sum_{n=0}^{\infty} \left[ \frac{F_o}{\pi n} \sin\left( n2\pi \frac{t_o}{T_o} \right) \cos(n\omega_o t + \angle H(n\omega_o)) + \frac{F_o}{\pi n} \left[ \cos\left( n2\pi \frac{t_o}{T_o} \right) - 1 \right] \sin(n\omega_o t + \angle H(n\omega_o)) \right]
\]

where

\[
H(n\omega_o) = \frac{1}{K - M(n\omega_o)^2 + j n\omega_o C}
\]

and is plotted in the figure below for different values of \( n \). Note that the response approximation does not become much more accurate beyond \( n=2 \) terms in the Fourier series because the building attenuates higher frequencies and only allows lower frequencies to pass into the response.
PROBLEM 3

Find and graph (by hand) the frequency response function between the excitation, \( f(t) \), and the response, \( x(t) \), for the system below. Compare these results qualitatively with the frequency response results for a standard single degree-of-freedom viscously underdamped oscillator. What effect does the series damper-spring have on the frequency response? This system is often used to model elastomer mounts and bushings.

The equation of motion for the harmonically forced system above with an elastomer support is given below:

\[
M\ddot{x} + C\left(\dot{x} - \dot{x}_o\right) + Kx = F_i \sin \omega t \\
C\left(\dot{x} - \dot{x}_o\right) = K_e x_o
\]

Since the excitation of the imaginary part of the rotating phasor, \( F_i e^{j\omega t} \), then both steady state responses, \( x_p(t) \) and \( x_{op}(t) \) can also be written as the imaginary parts of the phasors, \( X_p e^{j\omega t} \) and \( X_{op} e^{j\omega t} \), where both \( X_p \) and \( X_{op} \) are typically complex numbers. When these phasors are substituted into the equations above, we can solve them simultaneously by placing them in matrix form as follows:

\[
\begin{bmatrix}
K - M\omega^2 & j\omega C \\
-j\omega C & j\omega C + K_e
\end{bmatrix}
\begin{bmatrix}
X_p \\
X_{op}
\end{bmatrix} =
\begin{bmatrix}
F_i \\
0
\end{bmatrix}
\]

Now, we can solve for \( X_{op} \) in the second equation and substitute the result into the first equation in order to obtain the following form:
We note from the equation above that stiffness does not seem to have been affected by the elastomer, but damping has been affected. In fact, the damping coefficient appears to be dependent on frequency according to:

\[
C_{\text{effective}} = C - \frac{j\omega C^2}{j\omega C + K_e}
\]

This means that for low frequency, the damping is approximately \( C \), for high frequencies the damping is approximately zero, and for frequencies in between the damping is in between these values. The frequency response for this type of system is shown below and in the course notes. Note that for large values of the secondary stiffness, \( K_e \), the FRF approaches that of a viscously damped system.
It is clear from this plot that the damping is actually decreasing for increasing frequency. The transmissibility characteristic for an elastomer support is shown below (and also in the course notes). Note that the damping is effectively decreasing to zero for large frequencies, which, as we will soon find out, means that the system is better suited as an isolation system than a corresponding viscously damped isolation pad.
PROBLEM 4:

Find and plot (by hand) the approximate frequency response function of the single degree of freedom system below that is subjected to Coulomb friction and a simple harmonic input, \( F_i \cos(\omega t) \). Assume the damping force, \( \mu N \text{sgn}(dx/dt) \), is small compared to the excitation amplitude, \( F_i \), so that the steady state response can be assumed to be harmonic at the excitation frequency. Recall that the energy dissipated by Coulomb friction per cycle of harmonic oscillation at a frequency \( \omega \) is \( 4\mu NX_p \), where \( \mu \) is the static coefficient of friction, \( N \) is the normal force, and \( X_p \) is the steady state response amplitude at \( \omega \). Find the analytical condition on the friction force such that this solution is valid.

We use an equivalent viscous damping approach to solve the problem. Since the dissipation per cycle due to Coulomb damping is \( 4\mu NX_p \), then the equivalent viscous damping coefficient is found by equating this with the dissipation due to a viscous damper:

\[
\pi c_{eq} \omega X_p^2 = 4\mu NX_p
\]

\[
c_{eq} = \frac{4\mu N}{\pi \omega X_p}
\]

Then we can use the frequency response function for a SDOF with viscous damping given this value of the viscous damping coefficient.

\[
H(j\omega) = \frac{1}{K - M\omega^2 + j\omega c_{eq}}
\]

\[
\frac{X_p}{F_i} = \frac{1}{\sqrt{(K - M\omega^2) + \frac{16\mu^2 N^2 \omega^2}{\pi^2 \omega^2 X_p^2}}}
\]
which can be solved for the particular response amplitude of motion as follows:

\[ X_p^2 (K - M\omega^2) + \frac{16\mu^2 N^2 \omega^2}{\pi^2 \omega^2} = F_i^2 \]

\[ X_p = \sqrt{\frac{F_i^2 - \frac{16\mu^2 N^2}{\pi^2}}{(K - M\omega^2)}} \]

\[ \frac{X_p}{F_i / K} = \sqrt{1 - \left( \frac{4\mu N}{\pi F_i} \right)^2 \left( 1 - \left( \frac{\omega}{\omega_n} \right)^2 \right)} \]

This result shows that the amplitude is real if,

\[ 1 - \left( \frac{4\mu N}{\pi F_i} \right)^2 > 0 \]

\[ \frac{F_i}{\mu N} > \frac{4}{\pi} \]

When this condition holds, the response can be assumed to be primarily harmonic at the excitation frequency and the nonlinear damping can be approximated with linear viscous damping as we have done. Also note that the amplitude goes to infinity at the undamped resonant frequency unlike in the viscous damping case. This is because the relative phase between the excitation and the response at \( \omega_n \) is 90 degrees and because the denominator in the magnitude of the FRF goes to zero.

The phase of the FRF is given by the argument of the FRF:

\[ \phi_p = -\tan^{-1} \frac{4\mu N}{\pi X_p} \frac{X_p}{K - M\omega^2} = -\tan^{-1} \frac{4\mu N}{\pi K X_p} \frac{X_p}{1 - \left( \frac{\omega}{\omega_n} \right)^2} = -\tan^{-1} \frac{\pm 4\mu N}{\pi F_i} \frac{F_i}{\sqrt{1 - \left( \frac{4\mu N}{\pi F_i} \right)^2}} \]

Note that the relative phase is a constant (-7.3 degrees and -39.5 degrees for the two different values of the ratio). This result makes sense because as the forcing amplitude becomes much larger than the Coulomb friction force, the relative phase approaches zero (meaning there is no phase delay). Plots of the harmonic relative magnitude and phase for a system with Coulomb damping are given below. Note that the ratio of the Coulomb friction force to the excitation...
amplitude completely determines the shape of the FRF magnitude and the phase delay. It is interesting to note that for larger friction forces ($\mu N$), the amplitude at all frequencies is reduced by the exact same factor.

![Graph showing FRF magnitude and phase change for undamped system]