ME 563	HOMEWORK # 6 SOLUTIONS	Fall 2010

PROBLEM 1:

Assume that there is viscous damping distributed along a string as it vibrates. The damping coefficient per unit length is c. The density of the string is ρ, the tension is T, and the length is L. Derive the form of the free response, u(x,t), of the string assuming the string is fixed at both ends.

The equation of motion for such a string is:

$$\rho \frac{\partial^2 u}{\partial t^2} = T \frac{\partial^2 u}{\partial x^2} - c \frac{\partial u}{\partial t} \quad \text{where } u(0,t) = 0 = u(L,t)$$

The solution u(x,t)=G(t)*H(x) can then be substituted into this equation of motion yielding,

$$\rho H(x)\ddot{G} = TG(t)H'' - cH(x)\dot{G}$$

A separation of functions dependent on x and t, respectively, yields,

$$\rho \frac{\ddot{G}}{G} = T \frac{H''}{H} - c \frac{\dot{G}}{G}$$

$$\rho \frac{\ddot{G}}{G} + c \frac{\dot{G}}{G} = T \frac{H''}{H} = -\omega^2$$

In order for this expression to be true for all t and x, the following two conditions must be satisfied:

$$\ddot{G} + \frac{c}{\rho}\dot{G} + \omega^2 G = 0$$

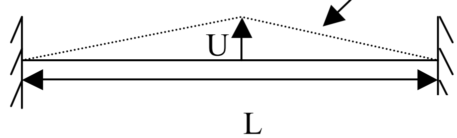
$$H'' + \frac{\omega^2}{c^2} H = 0 \text{ with } c^2 = T/\rho$$

After applying the boundary conditions, the form of the free response is found to be:

$$u(x,t) = \sum_{n=1}^{\infty} e^{-tc/2\rho}\left(A_n \cos\omega_{dn} t + B_n \sin\omega_{dn} t\right)\sin\frac{\omega_n}{c} x$$

$$\omega_{dn} = \sqrt{1 - \left(\frac{c}{2\rho\sqrt{nc\pi/L}}\right)^2}\, \frac{nc\pi}{L}$$

PROBLEM 2:
Assume a solution of the form found in Problem #1 with c=0. Given the triangular displacement initial condition shown below with zero initial velocity, compute the free response along the string. Plot the solution at t=0 using 2, 3, 10, and 20 modes of vibration to determine the accuracy of the response. Assume U=5 mm, L=1 m, and c=1000 m/s.



The solution is found by applying the initial conditions to the solution found above. Fourier series methods are used to carry out the calculations.

The functional form of the initial condition is given below:

$$u(x,0) = \begin{cases} \dfrac{2U}{L}x & x \le L/2 \\ U - \dfrac{2U}{L}\left(x - L/2\right) & x > L/2 \end{cases}$$

Because the initial velocity is zero for all x, B_n=0. The A_n are found by projecting the solution u(x,t) onto the initial displacement function. In other words, u(x,0) is written in terms of the series of modal deflections as follows:

$$u(x,0) = \sum_{n=1}^{\infty} A_n \cos\omega_n t \sin\frac{\omega_n}{c}x$$

The coefficients A_n are found by multiplying both sides of this equation by one modal deflection shape at a time, for mode m, and integrating by parts:

$$\int_0^L u(x,0)\sin\frac{\omega_m}{c}x dx = \int_0^L \sin\frac{\omega_m}{c}x \sum_{n=1}^{\infty} A_n \sin\frac{\omega_n}{c}x dx$$

Due to the orthogonality of sinusoidal functions at different frequencies, the right hand side of this equation reduces to:

$$\int_0^L u(x,0)\sin\frac{\omega_m}{c}x dx = \int_0^L A_m \sin^2\frac{\omega_m}{c}x dx$$

Then the integrals on both sides of the equation are computed (ouch…):

$$\int_0^L u(x,0)\sin\frac{\omega_m}{c}xdx = \int_0^{L/2} u(x,0)\sin\frac{\omega_m}{c}xdx + \int_{L/2}^L u(x,0)\sin\frac{\omega_m}{c}xdx$$

$$= \int_0^{L/2} \frac{2U}{L}x\sin\frac{\omega_m}{c}xdx + \int_{L/2}^L \left(U - \frac{2U}{L}\left(x - \frac{L}{2}\right)\right)\sin\frac{\omega_m}{c}xdx$$

$$= \frac{2U}{L}\int_0^{L/2} x\sin\frac{\omega_m}{c}xdx + \int_{L/2}^L \left(2U\sin\frac{\omega_m}{c}x - \frac{2U}{L}x\sin\frac{\omega_m}{c}x\right)dx$$

$$= \frac{2U}{L}\left[-\frac{c}{\omega_m}x\cos\frac{\omega_m}{c}x\Big|_0^{L/2} + \frac{c}{\omega_m}\int_0^{L/2}\cos\frac{\omega_m}{c}xdx\right] + \ldots$$

$$= \frac{2U}{L}\left[-\frac{cL}{m\pi c}\frac{L}{2}\cos\frac{Lm\pi c}{L2c} + \frac{c}{\omega_m}\int_0^{L/2}\cos\frac{\omega_m}{c}xdx\right] + \ldots$$

$$= \frac{2U}{L}\left[-\frac{L^2}{2m\pi}\cos\frac{m\pi}{2} + \frac{c}{\omega_m}\int_0^{L/2}\cos\frac{\omega_m}{c}xdx\right] + \ldots$$

$$= \frac{2U}{L}\left[-\frac{L^2}{2m\pi}\cos\frac{m\pi}{2} + \frac{c^2}{\omega^2{}_m}\sin\frac{m\pi c}{cL}x\Big|_0^{L/2}\right] + \ldots$$

$$= \frac{2U}{L}\left[-\frac{L^2}{2m\pi}\cos\frac{m\pi}{2} + \frac{c^2}{\omega^2{}_m}\sin\frac{m\pi}{2}\right] + \ldots$$

$$= \frac{2U}{L}\left[-\frac{L^2}{2m\pi}\cos\frac{m\pi}{2} + \frac{c^2}{\omega^2{}_m}\sin\frac{m\pi}{2}\right] + \int_{L/2}^L\left(2U\sin\frac{\omega_m}{c}x - \frac{2U}{L}x\sin\frac{\omega_m}{c}x\right)dx$$

$$= \ldots - \frac{2Uc}{\omega_m}\cos\frac{m\pi c}{cL}x\Big|_{L/2}^L - \frac{2U}{L}\int_{L/2}^L x\sin\frac{\omega_m}{c}xdx$$

$$= \ldots - \frac{2LU}{m\pi}\cos\frac{m\pi}{L}x\Big|_{L/2}^L - \frac{2U}{L}\int_{L/2}^L x\sin\frac{\omega_m}{c}xdx$$

$$= \ldots - \frac{2LU}{m\pi}\left(\cos m\pi - \cos\frac{m\pi}{2}\right) - \frac{2U}{L}\int_{L/2}^L x\sin\frac{\omega_m}{c}xdx$$

$$= \ldots - \frac{2U}{L}\int_{L/2}^L x\sin\frac{\omega_m}{c}xdx$$

$$= \ldots - \frac{2U}{L}\left[-\frac{c}{\omega_m}x\cos\frac{\omega_m}{c}x\Big|_{L/2}^L + \frac{c}{\omega_m}\int_{L/2}^L\cos\frac{\omega_m}{c}xdx\right]$$

$$= \ldots - \frac{2U}{L}\left[-\frac{c}{\omega_m}L\cos\frac{\omega_m}{c}L + \frac{cL}{2\omega_m}\cos\frac{L\omega_m}{2c} + \frac{c}{\omega_m}\int_{L/2}^L\cos\frac{\omega_m}{c}xdx\right]$$

$$= \ldots - \frac{2U}{L}\left[-\frac{L^2}{m\pi}\cos m\pi + \frac{L^2}{2m\pi}\cos\frac{m\pi}{2} + \frac{c}{\omega_m}\int_{L/2}^L\cos\frac{\omega_m}{c}xdx\right]$$

$$= .. - \frac{2U}{L}\left[-\frac{L^2}{m\pi}\cos m\pi + \frac{L^2}{2m\pi}\cos\frac{m\pi}{2} + \frac{c}{\omega_m}\int_{L/2}^L\cos\frac{\omega_m}{c}xdx\right]$$

$$= .. - \frac{2U}{L}\left[-\frac{L^2}{m\pi}\cos m\pi + \frac{L^2}{2m\pi}\cos\frac{m\pi}{2} + \frac{c^2}{\omega^2{}_m}\left[\sin m\pi - \sin\frac{m\pi}{2}\right]\right]$$

$$= .. - \frac{2U}{L}\left[-\frac{L^2}{m\pi}\cos m\pi + \frac{L^2}{2m\pi}\cos\frac{m\pi}{2} - \frac{c^2}{\omega^2{}_m}\sin\frac{m\pi}{2}\right]$$

$$\int_0^L A_m \sin^2 \frac{\omega_m}{c} x dx = A_m \int_0^L \left(\frac{1}{2} - \cos 2\frac{\omega_m}{c} x \right) dx$$

$$= \frac{A_m L}{2}$$

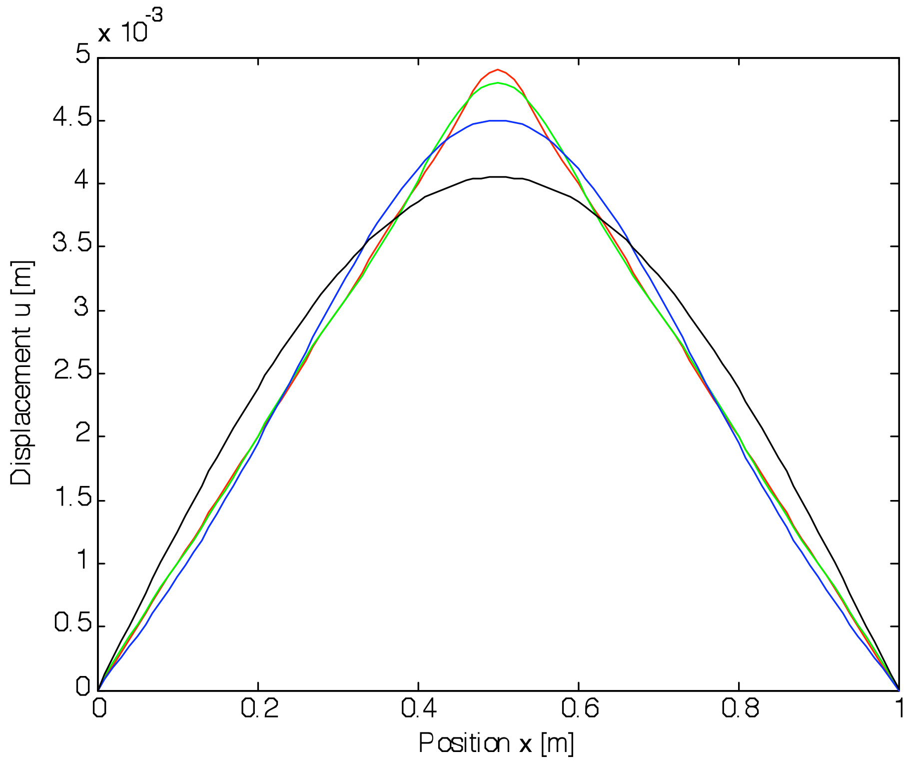
$$\frac{A_m L}{2} = \frac{2U}{L}\left[-\frac{L^2}{2m\pi}\cos\frac{m\pi}{2} + \frac{c^2}{\omega^2{}_m}\sin\frac{m\pi}{2} \right] - \frac{2LU}{m\pi}\left(\cos m\pi - \cos\frac{m\pi}{2} \right)$$

$$- \frac{2U}{L}\left[-\frac{L^2}{m\pi}\cos m\pi + \frac{L^2}{2m\pi}\cos\frac{m\pi}{2} - \frac{c^2}{\omega^2{}_m}\sin\frac{m\pi}{2} \right]$$

$$A_m = \frac{4U}{L^2}\left[-\frac{L^2}{2m\pi}\cos\frac{m\pi}{2} + \frac{c^2}{\omega^2{}_m}\sin\frac{m\pi}{2} \right] - \frac{4U}{m\pi}\left(\cos m\pi - \cos\frac{m\pi}{2} \right)$$

$$- \frac{4U}{L^2}\left[-\frac{L^2}{m\pi}\cos m\pi + \frac{L^2}{2m\pi}\cos\frac{m\pi}{2} - \frac{c^2}{\omega^2{}_m}\sin\frac{m\pi}{2} \right]$$

Then the solution can be plotted at t=0 to determine its accuracy for 2, 3, 10, and 20 values of m.



Black is for 2, blue is for 3, green is for 10, and red is for 20 modes of vibration to expand the solution. Note that we really only need three modes to accurately describe the response everywhere except the center of the string.