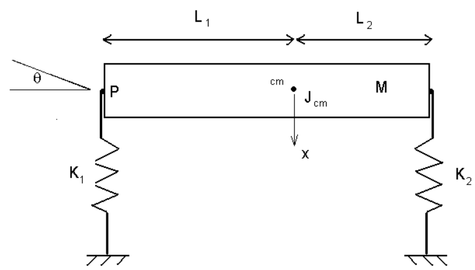
ME 563 **HOMEWORK # 5 SOLUTIONS** Fall 2010

PROBLEM 1:

You are given the lumped parameter dynamic differential equations of motion for a two degree-of-freedom model of an automobile suspension system for small rotations,

$$M\ddot{x} = -K_1(x - L_1\theta) - K_2(x + L_2\theta)$$
$$J_{cm}\ddot{\theta} = K_1(x - L_1\theta)L_1 - K_2(x + L_2\theta)L_2$$

with M=1,800 kg, L_1+L_2=3.6 m, L_1=1.4 m, K_1=42 kN/m, and K_2=48 kN/m and a radius of gyration of R=1.4 m.



Determine the modal frequencies, undamped natural frequencies of oscillation, and the modal vectors.

The modal frequencies for this system can be found from the corresponding eigenvalue problem, which is given by:

$$-\begin{bmatrix} M & 0 \\ 0 & J_{cm} \end{bmatrix}^{-1} \begin{bmatrix} K_1 + K_2 & K_2L_2 - K_1L_1 \\ K_2L_2 - K_1L_1 & K_1L_1^2 + K_2L_2^2 \end{bmatrix} \begin{Bmatrix} X \\ \Theta \end{Bmatrix} = \lambda \begin{Bmatrix} X \\ \Theta \end{Bmatrix} \text{ where } \lambda = s^2$$

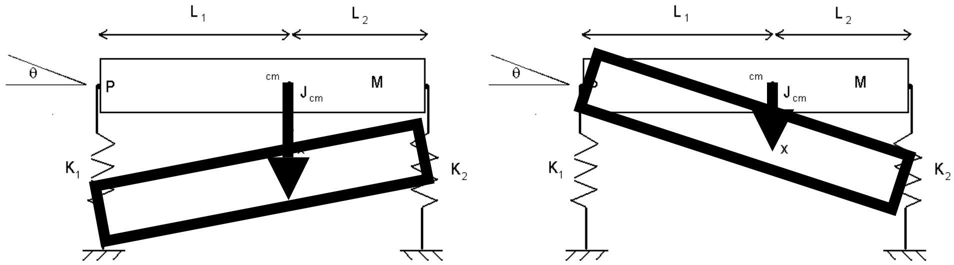
which can be solved in MATLAB to find the following eigenvalues, modal frequencies, undamped natural frequencies and eigenvectors (modal vectors):

$$\lambda_1 = -42.6; \lambda_2 = 96.6 \quad s_{1,2} = \pm j6.5 \text{ rad/s}; s_{3,4} = \pm j9.8 \text{ rad/s} \quad \omega_{n1} = 6.5 \text{ rad/s}; \omega_{n2} = 9.8 \text{ rad/s}$$

$$\begin{Bmatrix} X \\ \Theta \end{Bmatrix}_1 = \begin{Bmatrix} +1.0 \\ -0.3 \end{Bmatrix}; \begin{Bmatrix} X \\ \Theta \end{Bmatrix}_2 = \begin{Bmatrix} +0.6 \\ +1.0 \end{Bmatrix}$$

Draw schematics of the two modal vectors.

Schematics of these two modal vectors are shown below.



Also give the forms for the two principal modes of vibration.

The forms for the two principal modes of vibration each contain a temporal principal coordinate and a spatial modal vector. These forms are given below:

$$A_1 \begin{Bmatrix} +1.0 \\ -0.3 \end{Bmatrix} \sin(6.5t + \phi_1) \qquad A_2 \begin{Bmatrix} +0.6 \\ +1.0 \end{Bmatrix} \sin(9.8t + \phi_2)$$

Why are these modes adequate for describing general motions of the suspension system? These two modes are adequate because the principle of superposition holds for linear vibrating systems. When the differential equations of motion are re-derived in terms of the motion at point P instead of CM, which can be done relatively easily by substituting $x_p = x - L_1\theta$, although the modal vectors are different due to the coordinate transformation, the modal frequencies are identical. The reason that the natural frequencies do not change when the coordinates are changed is because the natural frequencies are properties of the system and do not depend on the coordinates selected.

<u>PROBLEM 2:</u>

Find and plot the translational and rotational responses of the suspension system above for initial conditions $x(0)$=0, $dx/dt(0)$=0.01 m/s, $\theta(0)$=0 rad, and $d\theta/dt(0)$=0.2 rad/s.

The translational and rotational responses of the suspension are found by substituting the ICs into the general form of the free response solution:

$$\begin{Bmatrix} x \\ \theta \end{Bmatrix} = A_1 \begin{Bmatrix} +1.0 \\ -0.3 \end{Bmatrix} \sin(6.5t + \phi_1) + A_2 \begin{Bmatrix} +0.6 \\ +1.0 \end{Bmatrix} \sin(9.8t + \phi_2)$$

$$\begin{Bmatrix} x(0) \\ \theta(0) \end{Bmatrix} = \begin{Bmatrix} +1.0 \\ -0.3 \end{Bmatrix} A_1 \sin(\phi_1) + \begin{Bmatrix} +0.6 \\ +1.0 \end{Bmatrix} A_2 \sin(\phi_2) = \begin{bmatrix} +1.0 & +0.6 \\ -0.3 & +1.0 \end{bmatrix} \begin{Bmatrix} A_1 \sin(\phi_1) \\ A_2 \sin(\phi_2) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

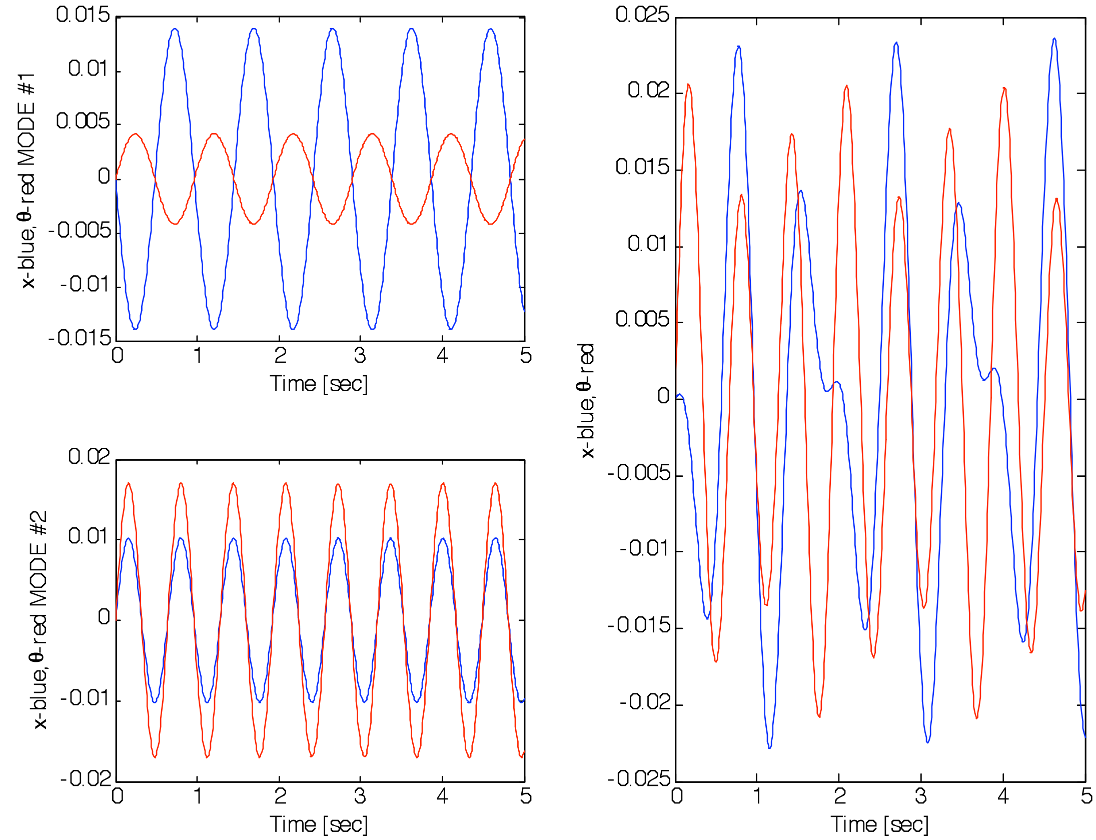
$$\Rightarrow \phi_1 = 0 = \phi_2 \text{ for nontrivial solutions}$$

$$\begin{Bmatrix} \dot{x}(0) \\ \dot{\theta}(0) \end{Bmatrix} = 6.5 \begin{Bmatrix} +1.0 \\ -0.3 \end{Bmatrix} A_1 \cos(\phi_1) + 9.8 \begin{Bmatrix} +0.6 \\ +1.0 \end{Bmatrix} A_2 \cos(\phi_2) = \begin{bmatrix} 6.5 \begin{pmatrix} +1.0 \\ -0.3 \end{pmatrix} & 9.8 \begin{pmatrix} +0.6 \\ +1.0 \end{pmatrix} \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = \begin{Bmatrix} 0.01 \\ 0.2 \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = \begin{Bmatrix} -0.014 \\ 0.017 \end{Bmatrix}$$

$$\therefore \begin{Bmatrix} x \\ \theta \end{Bmatrix} = -0.014 \begin{Bmatrix} +1.0 \\ -0.3 \end{Bmatrix} \sin(6.5t) + 0.017 \begin{Bmatrix} +0.6 \\ +1.0 \end{Bmatrix} \sin(9.8t)$$

Plot and label the two individual modal responses in addition to the total responses. These plots are shown below.



What should the initial conditions be in order not to excite the body pitch mode of vibration? To actually drive θ to zero for all time, the initial conditions would need to be chosen to satisfy

$$-0.3A_1\sin(6.5t+\phi_1)+1.0A_2\sin(9.8t+\phi_2)=0 \text{ for all } t$$

The only way to achieve this result is to make A1 and A2, both constants, equal to zero, which is the trivial solution. Having said that, one can make the pitch motion *as small as possible* by aligning the initial conditions with those of the first mode of vibration, which has far less contribution from the pitch motion than the bounce motion. In this case, we would choose:

$$\begin{Bmatrix} x(0) \\ \theta(0) \end{Bmatrix} = \begin{Bmatrix} +1.0 \\ -0.3 \end{Bmatrix}, \begin{Bmatrix} \dot{x}(0) \\ \dot{\theta}(0) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

What about the body bounce mode of vibration? The same is true for the bounce mode of vibration – we cannot make the bounce motion go to zero completely in the free response because there is bounce motion in both modes of vibration. We would need to choose the initial conditions aligned with the second mode, which involves primarily pitch motion. In some types of vibrating systems, however, the initial conditions can be chosen such that a given motion is not excited whatsoever (i.e., *A1* or *A2* is zero but not both).

PROBLEM 3:

How are the eigenvalues and eigenvectors of the system,

$$\begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} C_{11} & -C_{12} \\ -C_{12} & C_{22} \end{bmatrix}\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} 2K & -K \\ -K & 2K \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

with proportional damping related to its modal frequencies and modal vectors? You must formulate the eigen-problem for this set of equations in the state-space form. Assume that M=10 kg, C_{ij}=0.1M_{ij}+0.1K_{ij}, and K=10 N/m.

The equations above can be rewritten in state space form as follows:

$$\frac{d}{dt}\begin{Bmatrix} x_1 \\ x_2 \\ \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{2K}{M} & \frac{K}{M} & -\frac{C_{11}}{M} & \frac{C_{12}}{M} \\ \frac{K}{M} & -\frac{2K}{M} & \frac{C_{12}}{M} & -\frac{C_{22}}{M} \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \\ \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix}$$

With the given values for mass, stiffness, and damping, the eigenvalues are given by:

M=[10 0;0 10]; K=[20 -10;-10 20]; C=.1*M+.1*K; [v,d]=eig([zeros(2) eye(2); -inv(M)*K -inv(M)*C])

d=

-0.2000 + 1.7205i	**0**	**0**	**0**
0	**-0.2000 - 1.7205i**	**0**	**0**
0	**0**	**-0.1000 + 0.9950i**	**0**
0	**0**	**0**	**-0.1000 - 0.9950i**

And the eigenvectors are:

v =

0.0408 + 0.3512i	**0.0408 - 0.3512i**	**0.0500 + 0.4975i**	**0.0500 - 0.4975i**
-0.0408 - 0.3512i	**-0.0408 + 0.3512i**	**0.0500 + 0.4975i**	**0.0500 - 0.4975i**
-0.6124 + 0.0000i	**-0.6124 - 0.0000i**	**-0.5000 - 0.0000i**	**-0.5000 + 0.0000i**
0.6124	**0.6124**	**-0.5000**	**-0.5000**

These eigenvalues are identical to the solutions *s* of our characteristic equation. The entries of the eigenvectors corresponding to the two displacement states (first two states) have the same relative amplitudes (and phases) as the modal vectors we found. We call them real normal modes because they can be scaled to be all real and are the same as the undamped modal vectors.

v(:,1)/v(1,1)

ans =

```
   1.0000
  -1.0000 - 0.0000i
  -0.2000 + 1.7205i
   0.2000 - 1.7205i
```

v(:,3)/v(1,3)

ans =

```
   1.0000
   1.0000 - 0.0000i
  -0.1000 + 0.9950i
  -0.1000 + 0.9950i
```

PROBLEM 4:

You are given the lumped parameter linear differential equations of motion for a two degree-of-freedom model of a mechanical system:

$$\begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 8 & -5 \\ -5 & 6 \end{bmatrix}\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} 2K & -K \\ -K & 2K \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

with M=10 kg and K=10 N/m. Determine the modal frequencies, undamped natural frequencies of oscillation, and the modal vectors. Draw schematics of the two modal vectors. How are these modal vectors different when C=0 N-s/m? Draw the modes in this case for comparison.

The same approach as above yields:

M=[10 0;0 10]; K=[20 -10;-10 20]; C=[8 -5;-5 6]; [v,d]=eig([zeros(2) eye(2); -inv(M)*K -inv(M)*C])

v =

```
  -0.1278 - 0.3448i  -0.1278 + 0.3448i  -0.0907 - 0.4822i  -0.0907 + 0.4822i
   0.1619 + 0.2989i   0.1619 - 0.2989i  -0.0504 - 0.5056i  -0.0504 + 0.5056i
   0.6356             0.6356             0.4898 - 0.0425i   0.4898 + 0.0425i
  -0.5817 + 0.0828i  -0.5817 - 0.0828i   0.5092             0.5092
```

d =

```
  -0.6007 + 1.6205i        0                  0                  0
        0           -0.6007 - 1.6205i         0                  0
        0                  0           -0.0993 + 0.9972i         0
        0                  0                  0           -0.0993 - 0.9972i
```

Therefore,

$$\sigma_1 = -0.6007 r/s$$
$$\sigma_2 = -0.0993 r/s$$
$$\omega_{d1} = 1.6205 r/s$$
$$\omega_{d2} = 0.9972 r/s$$

And the same scaling rules for the vectors lead to,

v(:,1)/v(1,1)

ans =

```
   1.0000
  -0.9152 + 0.1303i
  -0.6007 + 1.6205i
   0.3385 - 1.5614i
```

v(:,3)/v(1,3)

ans =

```
   1.0000
   1.0318 + 0.0896i
  -0.0993 + 0.9972i
  -0.1919 + 1.0201i
```

It is evident that these modal vectors are NOT real normal modes because they cannot be scaled such that they become entirely real. The relatively small imaginary parts (.1303i and .0896i) make the modal vectors somewhat complex, which causes a "waviness" to the mode shape. In other words, the degrees of freedom do not move completely in or out of phase with one another. The first two eigenvectors lead to the same modal vectors because we must use a wrapped phase (modulus 180 deg); therefore, in the first column eigenvector, the second degree of freedom leads the first degree of freedom by nearly 180 deg, whereas in the second column, the first degree of freedom lags the second degree of freedom by nearly 180 deg by the same amount.

When the damping goes to zero, the modes become real normal modes and are the same as for the case with proportional damping.