ME 563 HOMEWORK # 3 SOLUTIONS Fall 2008

Problem 1

Assumptions

- Gravity acting downwards.

Lagrange's equation

Potential and kinetic energy expressions

$\vec{r}_{om1} = a.sin\theta_1\hat{\imath} - a.cos\theta_1\hat{\jmath}$

$\dot{\vec{r}}_{om1} = a\dot{\theta}_1cos\theta_1\hat{\imath} + a\dot{\theta}_1sin\theta_1\hat{\jmath}$

$\vec{r}_{om2} = ((a+b)sin\theta_1 + c.sin\theta_2)\hat{\imath} - ((a+b)cos\theta_1 + c.cos\theta_2)\hat{\jmath}$

$\dot{\vec{r}}_{om2} = \left((a+b)\dot{\theta}_1cos + c\dot{\theta}_2cos\theta_2\right)\hat{\imath} + ((a+b)\dot{\theta}_1sin\theta_1 + \dot{\theta}_2c.sin\theta_2)\hat{\jmath}$

$$T = \frac{1}{2}M_1\dot{\vec{r}}_{om1}.\dot{\vec{r}}_{om1} + \frac{1}{2}M_2\dot{\vec{r}}_{om2}.\dot{\vec{r}}_{om2} + \frac{1}{2}I_{om1}\dot{\theta}_1^2 + \frac{1}{2}I_{om2}\dot{\theta}_2^2$$

$$T = \frac{1}{2}M_1[(a\dot{\theta}_1cos\theta_1)^2 + (a\dot{\theta}_1sin\theta_1)^2] + \frac{1}{2}M_2\left[\left((a+b)\dot{\theta}_1cos\theta_1 + c\dot{\theta}_2cos\theta_2\right)^2 + ((a+b)\dot{\theta}_1sin\theta_1 + \dot{\theta}_2c.sin\theta_2)^2\right] + \frac{1}{2}I_{om1}\dot{\theta}_1^2 + \frac{1}{2}I_{om2}\dot{\theta}_2^2$$

$$T = \frac{1}{2}M_1(a\dot{\theta}_1)^2 + \frac{1}{2}M_2\left[(a+b)^2\dot{\theta}_1{}^2 + 2(a+b)c\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2) + \dot{\theta}_2{}^2c^2\right] + \frac{1}{2}I_{om1}\dot{\theta}_1^2 + \frac{1}{2}I_{om2}\dot{\theta}_2^2$$

$$V = -M_1ga.cos\theta_1 - [M_2g(a+b)\cos\theta_1 + M_2gc.cos\theta_2]$$

Generalized forces

$\vec{r}_{F1} = ((a+b)sin\theta_1)\hat{\imath} - ((a+b)cos\theta_1)\hat{\jmath}$

$\vec{r}_{F2} = ((a+b)sin\theta_1 + (c+d)sin\theta_2)\hat{\imath} - ((a+b)cos\theta_1 + (c+d)cos\theta_2)\hat{\jmath}$

$$Q_1^* = \vec{F}_1\frac{\partial\vec{r}_{F1}}{\partial\theta_1} + \vec{F}_2\frac{\partial\vec{r}_{F2}}{\partial\theta_1}$$

$$Q_1^* = F_1\hat{\imath}.[(a+b)cos\theta_1\hat{\imath} + (a+b)sin\theta_1\hat{\jmath}] + F_2\hat{\jmath}.[(a+b)cos\theta_1\hat{\imath} + (a+b)sin\theta_1\hat{\jmath}]$$

$$Q_1^* = F_1(a+b)cos\theta_1 + F_2(a+b)sin\theta_1$$

$$Q_2^* = \vec{F}_1\frac{\partial\vec{r}_{F1}}{\partial\theta_2} + \vec{F}_2\frac{\partial\vec{r}_{F2}}{\partial\theta_2}$$

$$Q_2^* = F_1\hat{\imath}.[0] + F_2\hat{\jmath}.[(c+d)cos\theta_2\hat{\imath} + (c+d)sin\theta_2\hat{\jmath}]$$

$$Q_2^* = F_2(c+d)sin\theta_2$$

Plugging all above into:

$$\frac{d}{dt}\left[\frac{\partial(T-V)}{\partial\dot{\theta}_1}\right] - \frac{\partial(T-V)}{\partial\theta_1} = Q_1^*$$

$$\Rightarrow \frac{d}{dt}[M_1a^2\dot{\theta}_1 + I_{om1}\dot{\theta}_1 + M_2(a+b)^2\dot{\theta}_1 + M_2(a+b)c\dot{\theta}_2\cos(\theta_1-\theta_2)]$$
$$- [-M_2(a+b)c\dot{\theta}_1\dot{\theta}_2\sin(\theta_1-\theta_2) - (M_1ga.sin\theta_1 + M_2g(a+b)\sin\theta_1)]$$
$$= F_1(a+b)cos\theta_1 + F_2(a+b)sin\theta_1$$

$$\Rightarrow M_1a^2\ddot{\theta}_1 + I_{om1}\ddot{\theta}_1 + M_2(a+b)^2\ddot{\theta}_1 + M_2(a+b)c\ddot{\theta}_2\cos(\theta_1-\theta_2) - M_2(a+b)c\dot{\theta}_1\dot{\theta}_2\sin(\theta_1-\theta_2) + M_2(a+b)c\dot{\theta}_2^{\,2}\sin(\theta_1-\theta_2) + M_2(a+b)c\dot{\theta}_1\dot{\theta}_2\sin(\theta_1-\theta_2) + M_1ga.sin\theta_1 + M_2g(a+b)\sin\theta_1$$
$$= F_1(a+b)cos\theta_1 + F_2(a+b)sin\theta_1$$

And plugging from above into:

$$\frac{d}{dt}\left[\frac{\partial(T-V)}{\partial\dot{\theta}_2}\right] - \frac{\partial(T-V)}{\partial\theta_2} = Q_2^*$$

$$\Rightarrow \frac{d}{dt}[I_{om2}\dot{\theta}_2 + M_2(a+b)c\dot{\theta}_1\cos(\theta_1-\theta_2) + M_2\dot{\theta}_2c^2]$$
$$- [M_2(a+b)c\dot{\theta}_1\dot{\theta}_2\sin(\theta_1-\theta_2) - M_2gcsin\theta_2] = F_2(c+d)sin\theta_2$$

$$\Rightarrow I_{om2}\ddot{\theta}_2 + M_2(a+b)c\ddot{\theta}_1\cos(\theta_1-\theta_2) - M_2(a+b)c\dot{\theta}_1^{\,2}\sin(\theta_1-\theta_2)$$
$$+ M_2(a+b)c\dot{\theta}_1\dot{\theta}_2sin(\theta_1-\theta_2) + M_2\ddot{\theta}_2c^2$$
$$- M_2(a+b)c\dot{\theta}_1\dot{\theta}_2\sin(\theta_1-\theta_2) - M_2gcsin\theta_2 = F_2(c+d)sin\theta_2$$

Problem 2

Assumptions

- Equilibrium position: $\theta_1 = \theta_2 = 0$
- New coordinate, $\theta_e = \theta - \Delta\theta$
- Small angles, i.e. $sin\theta \approx \theta$ and $cos\theta \approx \left[1 - \frac{\theta^2}{2}\right]$

Re-deriving linearized potential and kinetic energy expressions:

$$T = \frac{1}{2}M_1 a^2 \Delta\dot{\theta}_1^2 + \frac{1}{2}I_{om1}\Delta\dot{\theta}_1^2 + \frac{1}{2}I_{om2}\Delta\dot{\theta}_2^2 + \frac{1}{2}M_2\left[(a+b)^2\Delta\dot{\theta}_1^2 + c^2\Delta\dot{\theta}_2^2 + 2(a+b)c\Delta\dot{\theta}_1\Delta\dot{\theta}_2\right]$$

$$V = -M_1 ga\left[1 - \frac{\Delta\theta_1{}^2}{2}\right] - M_2 g(a+b)\left[1 - \frac{\Delta\theta_1{}^2}{2}\right] - M_2 gc\left[1 - \frac{\Delta\theta_2{}^2}{2}\right]$$

Generalized forces

$$Q_1^* = F_1(a+b)\left[1 - \frac{\Delta\theta_1{}^2}{2}\right] - F_2(a+b)\Delta\theta_1$$

$$Q_2^* = -F_2(c+d)\Delta\theta_2$$

Plugging into:

$$\frac{d}{dt}\left[\frac{\partial(T-V)}{\partial\dot{\theta}_1}\right] - \frac{\partial(T-V)}{\partial\theta_1} = Q_1^*$$

$$\Rightarrow M_1 a^2\Delta\ddot{\theta}_1 + I_{om1}\Delta\ddot{\theta}_1 + M_2(a+b)^2\Delta\ddot{\theta}_1 + M_2(a+b)c\Delta\ddot{\theta}_2 + M_1 ga\Delta\theta_1 + M_2 g(a+b)\Delta\theta_1 = F_1(a+b)$$

$$\frac{d}{dt}\left[\frac{\partial(T-V)}{\partial\dot{\theta}_2}\right] - \frac{\partial(T-V)}{\partial\theta_2} = Q_2^*$$

$$\Rightarrow I_{om2}\Delta\ddot{\theta}_2 + M_2 c^2\Delta\ddot{\theta}_2 + M_2(a+b)c\Delta\ddot{\theta}_1 + M_2 gc\theta_2 = 0$$

Problem 3

Assumptions

- Gravity acting vertically downwards.
- Cable is initially undeformed.

1) Deriving Lagrange's equations then linearizing.

$$\vec{r}_{om} = \frac{L}{2}cos\theta\,\hat{\imath} + \frac{L}{2}sin\theta\,\hat{\jmath}$$

$$\dot{\vec{r}}_{om} = -\frac{L}{2}\dot{\theta}sin\theta\,\hat{\imath} + \frac{L}{2}\dot{\theta}cos\theta\,\hat{\jmath}$$

$$\vec{r}_{Ma} = (Lcos\theta + rsin\phi)\hat{\imath} + (Lsin\theta - rcos\phi)\hat{\jmath}$$

$$\dot{\vec{r}}_{Ma} = (-L\dot{\theta}sin\theta + \dot{r}sin\phi + r\dot{\phi}cos\phi)\hat{\imath} + (L\dot{\theta}cos\theta - \dot{r}cos\phi + r\dot{\phi}sin\phi)\hat{\jmath}$$

$$T = \frac{1}{2}I_{om}\dot{\theta}^2 + \frac{1}{2}M\dot{\vec{r}}_{om}.\dot{\vec{r}}_{om} + \frac{1}{2}M_a\dot{\vec{r}}_{Ma}.\dot{\vec{r}}_{Ma}$$

$$T = \frac{1}{2}I_{om}\dot{\theta}^2 + \frac{1}{2}M[(\frac{L}{2}\dot{\theta})^2sin^2\theta + (\frac{L}{2}\dot{\theta})^2cos^2\theta] + \frac{1}{2}M_a[(-L\dot{\theta}sin\theta + \dot{r}sin\phi + r\dot{\phi}cos\phi)^2 + (L\dot{\theta}cos\theta - \dot{r}cos\phi + r\dot{\phi}sin\phi)^2]$$

$$T = \frac{1}{2}I_{om}\dot{\theta}^2 + \frac{1}{2}M\frac{L^2}{4}\dot{\theta}^2 + \frac{1}{2}M_a[L^2\dot{\theta}^2sin^2\theta - 2L\dot{\theta}\dot{r}sin\theta sin\phi - 2L\dot{\theta}\dot{\phi}rcos\phi sin\theta + \dot{r}^2sin^2\phi + 2\dot{r}r\dot{\phi}sin\phi cos\phi + r^2\dot{\phi}^2cos^2\phi + L^2\dot{\theta}^2cos^2\theta - 2L\dot{\theta}\dot{r}cos\theta cos\phi + 2L\dot{\theta}r\dot{\phi}sin\phi cos\theta + \dot{r}^2cos^2\phi - 2\dot{r}r\dot{\phi}cos\phi sin\phi + r^2\dot{\phi}^2sin^2\phi]$$

$$T = \frac{1}{2}I_{om}\dot{\theta}^2 + \frac{1}{2}M\frac{L^2}{4}\dot{\theta}^2 + \frac{1}{2}M_a[L^2\dot{\theta}^2 + \dot{r}^2 + r^2\dot{\phi}^2 - 2L\dot{\theta}\dot{r}\cos(\theta - \phi) + 2L\dot{\theta}r\dot{\phi}\sin(\phi - \theta)]$$

$$V = Mg\frac{L}{2}sin\theta + M_agLsin\theta - M_agrcos\,\phi + \frac{1}{2}K(r - r_u)^2$$

$$\frac{\partial V}{\partial \theta} = 0 = Mg\frac{L}{2}cos\theta + M_agLcos\theta \Rightarrow \theta_e = \pm\frac{m\pi}{2}$$

$$\frac{\partial V}{\partial \phi} = 0 = M_agrsin\phi \Rightarrow \phi_e = \pm n\pi$$

$$\frac{\partial V}{\partial r} = 0 = -M_agcos\,\phi + K(r - r_u) \Rightarrow r_e = \frac{M_ag}{K} + r_u$$

Choose equilibrium points:

$$\theta_e = \frac{\pi}{2}, \qquad \phi_e = \pi, \qquad r_e = \frac{M_ag}{K} + r_u$$

2) Deriving linearized EOMs directly.

$$\theta = \theta_e + \Delta\theta, \qquad \phi = \phi_e + \Delta\phi, \qquad r = r_e + \Delta r$$

$$T = \frac{1}{2} I_{om} \Delta\dot{\theta}^2 + \frac{1}{2} M (\frac{L^2}{4} \Delta\dot{\theta}^2) + \frac{1}{2} M_a [L^2 \Delta\dot{\theta}^2 + \Delta\dot{r}^2 + r_e{}^2 \Delta\dot{\phi}^2 - 2 L r_e \Delta\dot{\phi} \Delta\dot{\theta}]$$

$$V = Mg \frac{L}{2} \Delta\theta + M_a g L \Delta\theta - M_a g r_e \left[1 - \frac{\Delta\phi^2}{2}\right] + \frac{1}{2} K (\frac{M_a g}{K} + \Delta r)^2$$

$$\frac{d}{dt}\left[\frac{\partial (T - V)}{\partial \dot{\theta}}\right] - \frac{\partial (T - V)}{\partial \theta} = Q_\theta^*$$

$$\frac{d}{dt}[I_{om} \Delta\dot{\theta} + M \frac{L^2}{4} \Delta\dot{\theta} + M_a L^2 \Delta\dot{\theta} - M_a L r_e \Delta\dot{\phi}] + [Mg \frac{L}{2} + M_a g L] = \tau$$

$$I_{om} \Delta\ddot{\theta} + M \frac{L^2}{4} \Delta\ddot{\theta} + M_a L^2 \Delta\ddot{\theta} - M_a L r_e \Delta\ddot{\phi} + Mg \frac{L}{2} + M_a g L = \tau$$

$$\frac{d}{dt}\left[\frac{\partial (T - V)}{\partial \dot{\phi}}\right] - \frac{\partial (T - V)}{\partial \phi} = Q_\phi^*$$

$$\frac{d}{dt}\left(M_a r_e{}^2 \Delta\dot{\phi} - M_a L r_e \Delta\dot{\theta}\right) + M_a g r_e \Delta\phi = 0$$

$$M_a r_e{}^2 \Delta\ddot{\phi} - M_a L r_e \Delta\ddot{\theta} + M_a g r_e \Delta\phi = 0$$

$$\frac{d}{dt}\left[\frac{\partial (T - V)}{\partial \dot{r}}\right] - \frac{\partial (T - V)}{\partial r} = Q_r^*$$

$$\frac{d}{dt}(M_a \Delta\dot{r}) + K\left(\frac{M_a g}{K} + \Delta r\right) = 0$$

$$M_a \Delta\ddot{r} + K\left(\frac{M_a g}{K} + \Delta r\right) = 0$$