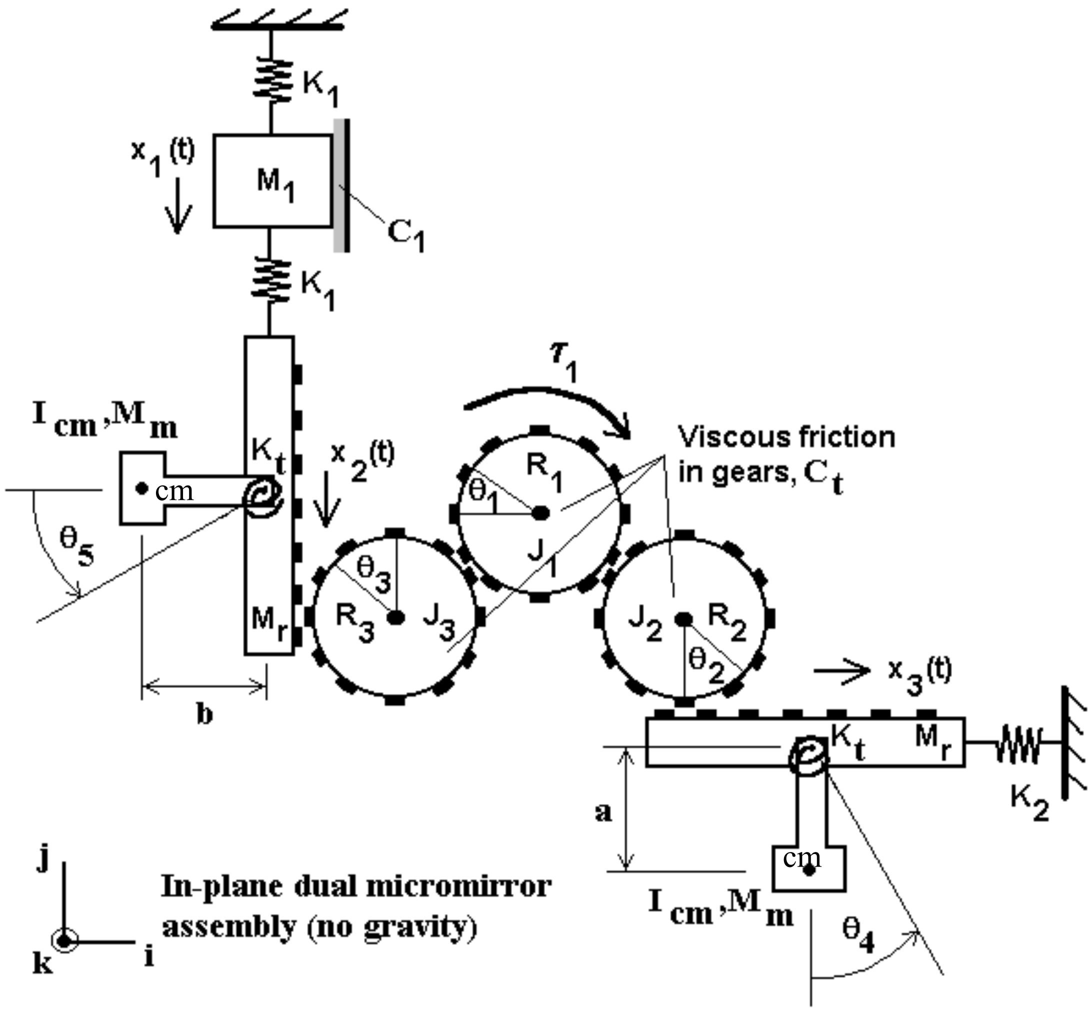
PROBLEM 1: (40%)

Derive the equations of motion for the dual micromirror system shown below using both Newton-Euler techniques and Lagrange's equations. Assume that the two racks, M_r, experience only translational motion. The two mirrors, I_{cm}, are free to swing about the pins that connect them to the two racks. Give all relevant information you need to solve the problem using the procedure developed in class.



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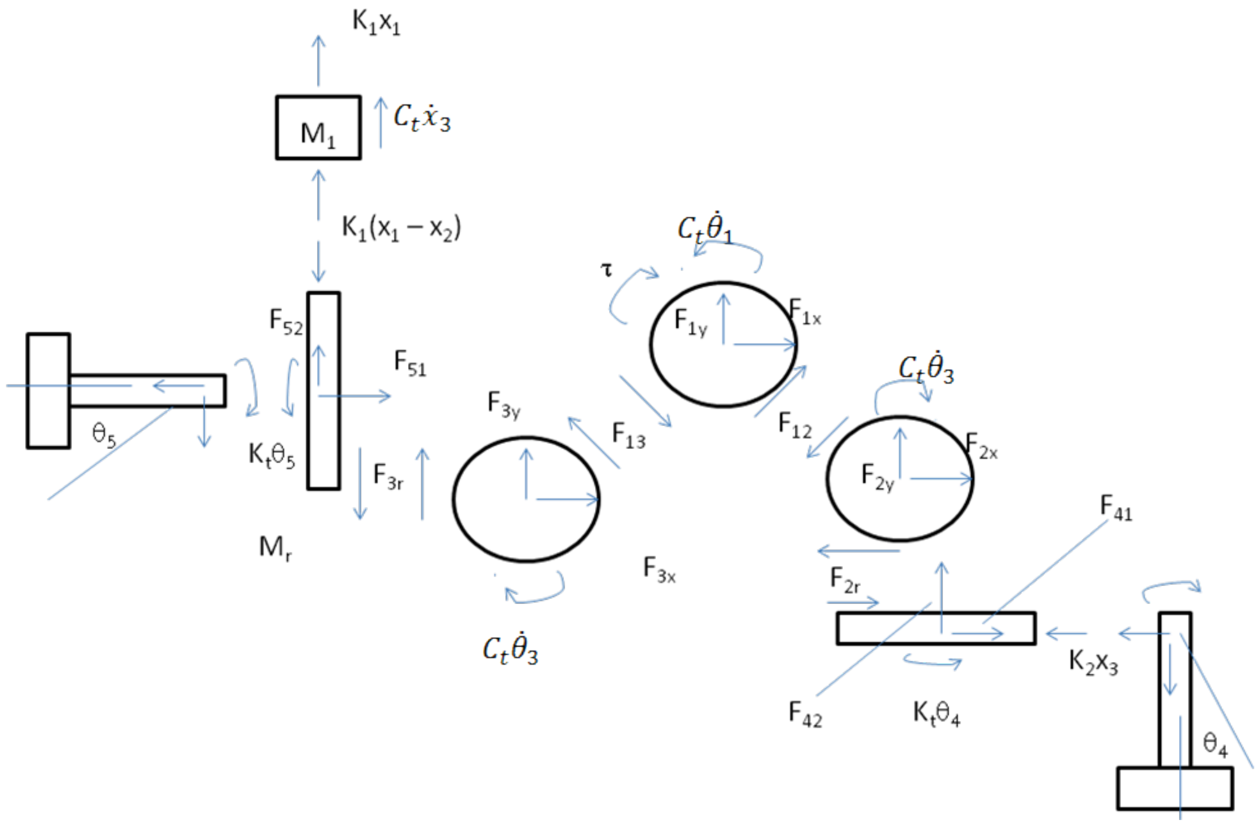


Assumptions

- No slip between gears, racks and pinions.
- Gravity ignored.
- Gear contacts have zero normal force.
- Friction modeled as viscous damping.
- Racks have one degree of motion.

Newton – Euler method

For M_1:

$$+\downarrow \sum F = M_1\ddot{x}_1 = -k_1x_1 - k_1(x_1 - x_2) - C_1\dot{x}_1$$

For M_{r2}:

$$+\downarrow \sum F = M_r\ddot{x}_2 = k_1(x_1 - x_2) - F_{5r} + F_{3r}$$

For I_{cm5}:

$$+CCW \sum M_{cm} = I_{cm}\ddot{\theta}_5 = -k_t\theta_5 - F_{52}bcos\theta_5$$

For J_3:

$$+CCW \sum M_{cm} = J_3\ddot{\theta}_3 = -C_t\dot{\theta}_3 - F_{3r}R_3 + F_{13}R_3$$

For J_1:

$$+CW \sum M_{cm} = J_1\ddot{\theta}_1 = -C_t\dot{\theta}_1 - F_{13}R_1 - F_{12}R_1 + \tau_1$$

For J_2:

$$+CCW \sum M_{cm} = J_2\ddot{\theta}_2 = -C_t\dot{\theta}_2 + F_{12}R_2 - F_{2r}R_2$$

For M_{r3}:

$C_t\dot{\theta}_2$

$$+\rightarrow \sum F = M_r\ddot{x}_3 = -k_2x_3 + F_{2r} + F_{41}$$

For I_{cm4}:

$$+CCW \sum M_{cm} = I_{cm}\ddot{\theta}_4 = -k_t\theta_4 + F_{41}acos\theta_4$$

No slip constraints:

$$R_3\theta_3 = R_1\theta_1$$
$$R_2\theta_2 = R_1\theta_1$$
$$x_2 = R_3\theta_3$$
$$x_3 = R_2\theta_2$$

Require 2 more equations since there are 14 unknowns and 12 equations.

For M_{m5}:

$$+\downarrow \sum F = M_m\ddot{x}_5 = F_{52}$$

For M_{m4} :

$$+\rightarrow \sum F = M_m\ddot{x}_4 = -F_{41}$$

where

$$x_5 = x_2 + b\ sin\theta_5$$
$$x_4 = x_3 + a\ sin\theta_4$$

There are now 14 equations and 14 unknowns, and x_1, θ_1, θ_4 and θ_5 can be found.

<u>Lagrange's equations</u>

$$T = \frac{1}{2}M_1\dot{x}_1^2 + \frac{1}{2}M_r\dot{x}_2^2 + \frac{1}{2}J_3\dot{\theta}_3^2 + \frac{1}{2}J_1\dot{\theta}_1^2 + \frac{1}{2}J_2\dot{\theta}_2^2 + \frac{1}{2}M_r\dot{x}_3^2 + \left[\frac{1}{2}I_{cm}\dot{\theta}_5^2 + \frac{1}{2}M_m\dot{\vec{r}}_5.\dot{\vec{r}}_5\right] + \left[\frac{1}{2}I_{cm}\dot{\theta}_4^2 + \frac{1}{2}M_m\dot{\vec{r}}_4.\dot{\vec{r}}_4\right]$$

$$V = \frac{1}{2}k_1x_1^2 + \frac{1}{2}k_1(x_2 - x_1)^2 + \frac{1}{2}k_t\theta_5^2 + \frac{1}{2}k_2x_3^2 + \frac{1}{2}k_t\theta_4^2$$

Rayleigh's dissipation function

$$R = \frac{1}{2}C_1\dot{x}_1^2 + \frac{1}{2}C_t\dot{\theta}_3^2 + \frac{1}{2}C_t\dot{\theta}_1^2 + \frac{1}{2}C_t\dot{\theta}_2^2$$

Compute partial and time derivatives

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_1} - \frac{\partial L}{\partial x_1} + \frac{\partial R}{\partial \dot{x}_1} = Q^{\star}_{x_1}$$

Etc.

Constraints are eliminated using:

$$R_3\theta_3 = R_1\theta_1$$
$$R_2\theta_2 = R_1\theta_1$$
$$x_2 = R_3\theta_3$$

$x_3 = R_2\theta_2$

We need the velocity of the center of mass of pendulums for Lagrange's equations

$$\vec{r_5} = -x_2 j - b\ sin\theta_5 j - b\ cos\theta_5 i$$
$$\dot{\vec{r_5}} = -\dot{x}_2 j - b\dot{\theta}_5\ cos\theta_5 j + b\dot{\theta}_5\ sin\theta_5 i$$

Therefore

$$\frac{1}{2}M_m\dot{\vec{r_5}}.\dot{\vec{r_5}} = \frac{1}{2}M_m\left[(b\dot{\theta}_5\ sin\theta_5)^2 + (\dot{x}_2 + b\dot{\dot{\theta}}_5\ cos\theta_{5_2} + b\dot{\theta}_5\ cos\theta_5)^2\right]$$
$$= \frac{1}{2}M_m\left[b^2\dot{\theta}_5{}^2\ sin^2\theta_5 + \dot{x}_2^2 + 2\dot{x}_2 b\dot{\theta}_5\ cos\theta_5 + b^2\dot{\theta}_5{}^2\ cos^2\theta_5\right]$$
$$= \frac{1}{2}M_m\left[b^2\dot{\theta}_5{}^2 + \dot{x}_2^2 + 2\dot{x}_2 b\dot{\theta}_5\ cos\theta_5\ \right]$$

Likewise for $\frac{1}{2}M_m\dot{\vec{r_4}}.\dot{\vec{r_4}}$

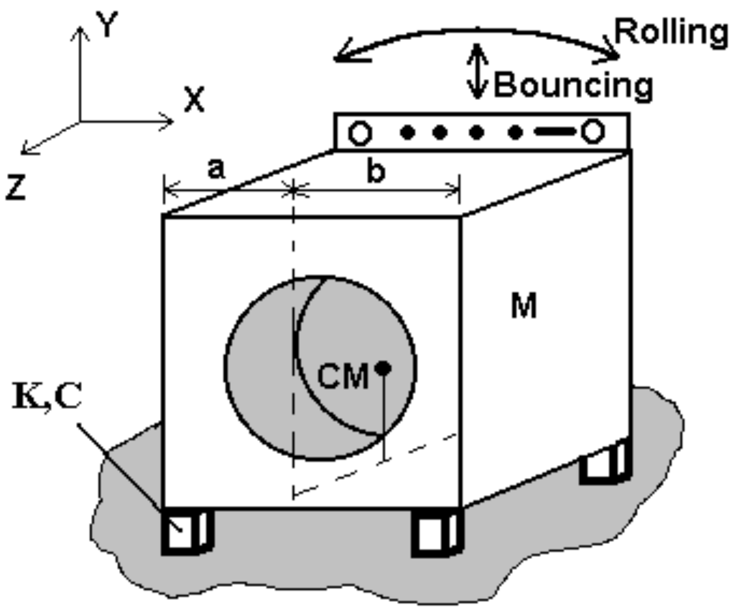
$$\vec{r_4} = +x_3 i - a\ cos\theta_4 j + a\ sin\theta_4 i$$
$$\dot{\vec{r_4}} = +\dot{x}_3 i + a\dot{\theta}_4\ sin\theta_4 j + a\dot{\theta}_4\ cos\theta_4 i$$

Hence

$$\frac{1}{2}M_m\dot{\vec{r_4}}.\dot{\vec{r_4}} = \frac{1}{2}M_m\left[(\dot{x}_3 + a\dot{\theta}_4\ cos\theta_4)^2 + (a\dot{\theta}_4\ sin\theta_4)^2\right]$$
$$= \frac{1}{2}M_m\left[a^2\dot{\theta}_4{}^2\ cos^2\theta_4 + \dot{x}_3^2 + 2\dot{x}_3 a\dot{\theta}_4\ cos\theta_4 + a^2\dot{\theta}_4{}^2\ sin^2\theta_4\right]$$
$$= \frac{1}{2}M_m\left[a^2\dot{\theta}_4{}^2 + \dot{x}_3^2 + 2\dot{x}_3 a\dot{\theta}_4\ cos\theta_4\ \right]$$

PROBLEM 2: (30%)

A clothes dryer of mass, M, is shown below with four rubber supports, which have stiffness K and viscous damping coefficient C. The unbalance effect can be modeled with a forcing function, *f(t)*, that is applied vertically through the geometric center of the basket as it rotates. Develop a mechanical vibration model of this system to describe the bouncing and rolling motions of the dryer for relatively small roll rotations, θ, and derive the equation(s) of motion using Lagrange's equations. Assume that the dryer supports are always in contact with the ground. Describe the effects of changes in a and b as well as changes in K and C for different supports.



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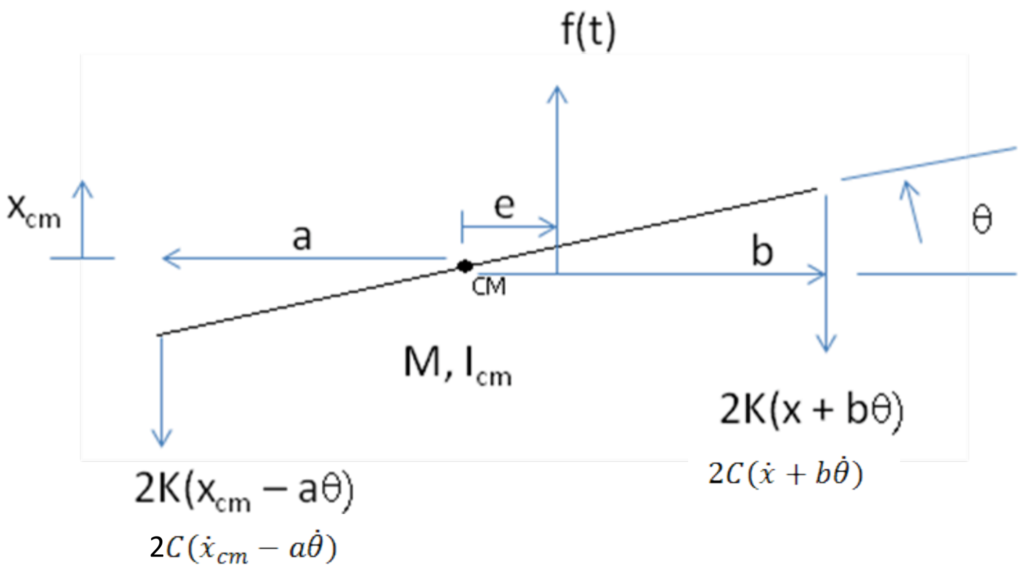


Assumptions

- Force f(t) is applied off the center of mass so vertical and rocking motions are produced.
- Small angles.
- Dryer supports always in contact with ground.
- e is moment arm for f(t).
- Deformations take place with respect to the equilibrium positions.

Equations of Motion

$$+\uparrow \sum F = M\ddot{x} = -2K(x_{cm} - a\theta) - 2K(x + b\theta) - 2C(\dot{x}_{cm} - a\dot{\theta}) - 2C(\dot{x} + b\dot{\theta}) + f(t)$$

$$+CCW \sum M_{cm} = I_{cm}\ddot{\theta}$$
$$= [2K(x_{cm} - a\theta) + 2C(\dot{x}_{cm} - a\dot{\theta})]a - [2K(x + b\theta) + 2C(\dot{x} + b\dot{\theta})]b + ef(t)$$

Lagrange's Equations

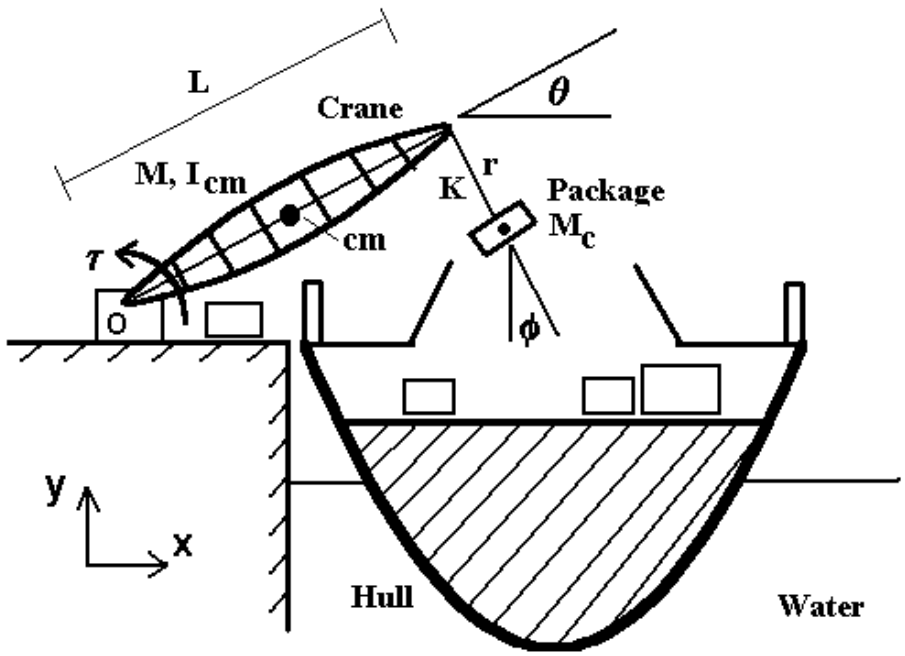
$$T = \frac{1}{2}M\dot{x}_{cm}^2 + \frac{1}{2}I_{cm}\dot{\theta}^2$$
$$V = \frac{1}{2}2K(x_{cm} - a\theta)^2 + \frac{1}{2}2K(x + b\theta)^2$$
$$R = \frac{1}{2}2C(\dot{x}_{cm} - a\dot{\theta})^2 + \frac{1}{2}2C(\dot{x} + b\dot{\theta})^2$$

Compute partial and time derivatives

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_{cm}} - \frac{\partial L}{\partial x_{cm}} + \frac{\partial R}{\partial \dot{x}_{cm}} = Q_{x_{cm}}^{\star}$$

PROBLEM 3 (30%)

A crane of mass, M, with mass moment of inertia, I_{cm}, is shown below moving a crate of mass, M_c. An applied torque, τ, at the base of the crane at point O positions the crane, which is a distance, r, away from the end of the crane. Assume that the cable has stiffness, K, which suggests that the cable can extend while remaining in tension at all times. Model the vibrations of the crane-crate system in the plane of the page and derive the equation(s) of motion using Lagrange's equations.



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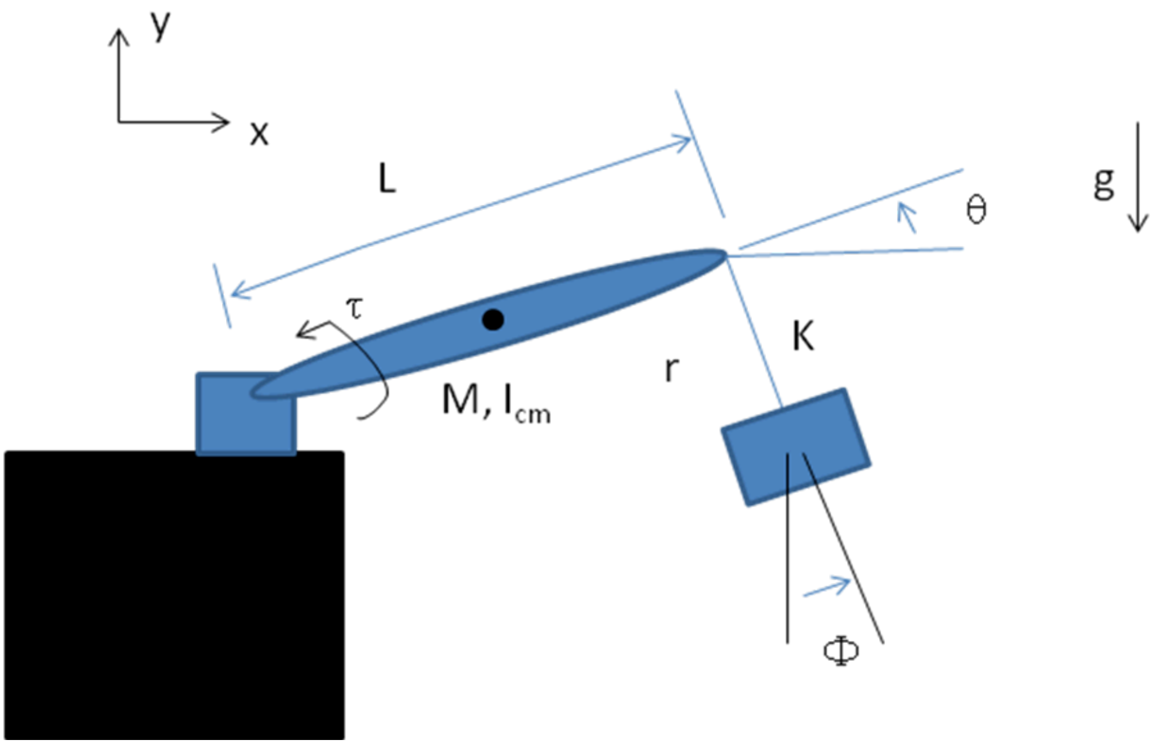

$$T = \frac{1}{2}M(\frac{L}{2}\dot{\theta})^2 + \frac{1}{2}I_{cm}\dot{\theta}^2 + \frac{1}{2}M_c\dot{\vec{r}_c}.\dot{\vec{r}_c}$$
$$V = \frac{1}{2}K(r - r_u)^2 + M_c g(Lsin\theta - rcos\phi) + Mg\frac{L}{2}sin\theta$$

Where

$$\vec{r_c} = (Lcos\theta + rsin\phi)i + (Lsin\theta - rcos\phi)j$$
$$\dot{\vec{r_c}} = (-Lsin\theta\dot{\theta} + \dot{r}sin\phi + r\dot{\phi}cos\phi)i + (Lcos\theta\dot{\theta} - \dot{r}cos\phi + r\dot{\phi}sin\phi)j$$

$$\frac{1}{2}M_c\dot{\vec{r}_c}.\dot{\vec{r}_c} = \frac{1}{2}M_c\left[\left(-Lsin\theta\dot{\theta} + \dot{r}sin\phi + r\dot{\phi}cos\phi\right)^2 + \left(Lcos\theta\dot{\theta} - \dot{r}cos\phi + r\dot{\phi}sin\phi\right)^2\right]$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = Q_r^\star \qquad Eq.(1)$$

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{r}} - \frac{\partial V}{\partial \dot{r}}\right] - \frac{\partial T}{\partial r} + \frac{\partial V}{\partial r} = Q_r^\star \qquad Eq.(1b)$$

$$\frac{\partial T}{\partial \dot{r}} = \frac{1}{2}M_c\left[2sin\phi\left(-Lsin\theta\dot{\theta} + \dot{r}sin\phi + r\dot{\phi}cos\phi\right) - 2cos\phi\left(Lcos\theta\dot{\theta} - \dot{r}cos\phi + r\dot{\phi}sin\phi\right)\right]$$
$$\frac{\partial T}{\partial r} = \frac{1}{2}M_c\left[2\dot{\phi}cos\phi\left(-Lsin\theta\dot{\theta} + \dot{r}sin\phi + r\dot{\phi}cos\phi\right) + 2\dot{\phi}sin\phi\left(Lcos\theta\dot{\theta} - \dot{r}cos\phi + r\dot{\phi}sin\phi\right)\right]$$

$$\frac{\partial V}{\partial \dot{r}} = 0$$
$$\frac{\partial V}{\partial r} = K(r - r_u) - M_c g\ cos\phi$$

$$Q_r^\star = 0$$

Plug all above into Eq. (1b)

$$\frac{d}{dt}\left\lfloor \frac{\partial T}{\partial \dot{\theta}} - \frac{\partial V}{\partial \dot{\theta}} \right\rfloor - \frac{\partial T}{\partial \theta} + \frac{\partial V}{\partial \theta} = Q_\theta^\star \qquad Eq.(2)$$
$$\partial T/(\partial \theta^{\cdot}\ ) = (M\theta^{\cdot} L\text{^}2)/4 + I\_cm\ \theta^{\cdot} + \quad 1/2\ M\_c\ \lfloor -2Lsin\theta(-Lsin\theta\theta^{\cdot} + r^{\cdot} sin\phi + r\phi^{\cdot} cos\phi) + 2Lcos\theta(Lcos\theta\theta^{\cdot} - r^{\cdot} cos\phi + r\phi^{\cdot} sin\phi)\ \rfloor$$
$$\frac{\partial T}{\partial \theta} = \frac{1}{2} M_c \lfloor -L\dot{\theta}cos\theta(-Lsin\theta\dot{\theta} + \dot{r}sin\phi + r\dot{\phi}cos\phi) - L\dot{\theta}sin\theta(Lcos\theta\dot{\theta} - \dot{r}cos\phi + r\dot{\phi}sin\phi) \rfloor$$
$$\frac{\partial V}{\partial \theta} = M_c g\ Lcos\theta + Mg\frac{L}{2}\cos\theta$$
$$Q_\theta^\star = \tau$$

Plug all above into Eq. (2)

$$\frac{d}{dt}\left\lfloor \frac{\partial T}{\partial \dot{\Phi}} - \frac{\partial V}{\partial \dot{\Phi}} \right\rfloor - \frac{\partial T}{\partial \Phi} + \frac{\partial V}{\partial \Phi} = Q_\Phi^\star \qquad Eq.(3)$$

$$\frac{\partial T}{\partial \dot{\Phi}} = \frac{1}{2} M_c \lfloor 2rcos\phi(-Lsin\theta\dot{\theta} + \dot{r}sin\phi + r\dot{\phi}cos\phi) - 2rsin\phi(Lcos\theta\dot{\theta} - \dot{r}cos\phi + r\dot{\phi}sin\phi) \rfloor$$

$$\frac{\partial T}{\partial \Phi} = \frac{1}{2} M_c \lfloor 2(\dot{r}cos\phi - r\dot{\phi}sin\phi)(-Lsin\theta\dot{\theta} + \dot{r}sin\phi + r\dot{\phi}cos\phi) + 2(\dot{r}sin\phi + r\dot{\phi}cos\phi)(Lcos\theta\dot{\theta} - \dot{r}cos\phi + r\dot{\phi}sin\phi) \rfloor$$

$$\frac{\partial V}{\partial \dot{\Phi}} = 0$$

$$\frac{\partial V}{\partial \Phi} = M_c g\ rsin\Phi$$

$$Q_\Phi^\star = 0$$

Plug all above into Eq. (3)